

1. 把 $\sin n\varphi$ 和 $\cos n\varphi$ 用 $\sin \varphi$ 和 $\cos \varphi$ 表示出来 (1.6)

$$\begin{aligned}\cos n\varphi + i \sin n\varphi &= (\cos \varphi + i \sin \varphi)^n = \sum_{k=0}^n \frac{n! i^k}{k!(n-k)!} \cos^{n-k} \varphi \sin^k \varphi \\&= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{(2k)!(n-2k)!} \cos^{n-2k} \varphi \sin^{2k} \varphi \\&\quad + i \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{n!}{(2k+1)!(n-2k-1)!} \cos^{n-2k-1} \varphi \sin^{2k+1} \varphi\end{aligned}$$

比较两边实部和虚部得：

$$\begin{aligned}\cos n\varphi &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{(2k)!(n-2k)!} \cos^{n-2k} \varphi \sin^{2k} \varphi ; \\ \sin n\varphi &= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{n!}{(2k+1)!(n-2k-1)!} \cos^{n-2k-1} \varphi \sin^{2k+1} \varphi .\end{aligned}$$

2. 将下列和式表示成有限形式：(1) $\sum_{k=1}^n \cos k\varphi$ ；(2) $\sum_{k=1}^n \sin k\varphi$ 。

$$\sum_{k=1}^n e^{ik\varphi} = e^{i\varphi} \frac{1 - e^{in\varphi}}{1 - e^{i\varphi}} = e^{i\varphi} \frac{e^{\frac{i n \varphi}{2}} \left(e^{-\frac{i n \varphi}{2}} - e^{\frac{i n \varphi}{2}} \right)}{e^{\frac{i \varphi}{2}} \left(e^{-\frac{i \varphi}{2}} - e^{\frac{i \varphi}{2}} \right)} = e^{\frac{i(n+1)\varphi}{2}} \frac{\sin \frac{n\varphi}{2}}{\sin \frac{\varphi}{2}}$$

比较两边实部和虚部得：

$$\sum_{k=1}^n \cos k\varphi = \frac{\sin \frac{n\varphi}{2} \cos \frac{(n+1)\varphi}{2}}{\sin \frac{\varphi}{2}}, \quad \sum_{k=1}^n \sin k\varphi = \frac{\sin \frac{n\varphi}{2} \sin \frac{(n+1)\varphi}{2}}{\sin \frac{\varphi}{2}}.$$

3. 证明： $\sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$ 。(1.8)

记 $1, z_1, z_2, \dots, z_{n-1}$ 为方程 $z^n = 1$ 的 n 个根，即 $z_k = e^{\frac{2k\pi}{n}}$ ， $k = 1, 2, \dots, n-1$ 。则有

$$z^n - 1 = (z-1)(z-z_1)(z-z_2) \cdots (z-z_{n-1}),$$

$$\text{所以 } (z-z_1)(z-z_2) \cdots (z-z_{n-1}) = \frac{z^n - 1}{z-1} = z^{n-1} + z^{n-2} + \cdots + z + 1.$$

$$\text{令上式两边 } z=1, \text{ 则有 } \prod_{k=1}^{n-1} \left(1 - e^{\frac{2k\pi}{n}} \right) = n.$$

$$1 - e^{\frac{2k\pi}{n}} = -e^{\frac{k\pi}{n}} \left(e^{\frac{k\pi}{n}} - e^{-\frac{k\pi}{n}} \right) = -2ie^{\frac{k\pi}{n}} \sin \frac{k\pi}{n} = 2e^{-\frac{\pi}{2}} e^{\frac{k\pi}{n}} \sin \frac{k\pi}{n},$$

$$\prod_{k=1}^{n-1} \left(1 - e^{\frac{2k\pi}{n}} \right) = 2^{n-1} e^{-\frac{(n-1)\pi}{2} + \sum_{k=1}^{n-1} \frac{k\pi}{n}} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = 2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n, \quad \text{即 } \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

二. 解析函数与C-R条件

4. 判断下列函数在何处可导（并求出导数），在何处解析：(1.10)

(1) $|z|$ ；(2) $\bar{z}$ ；(3) z^m ， $m=0, 1, 2, \dots$ ；(4) e^z ；(5) $(x^2+2y)+i(x^2+y^2)$ ；

(6) $(x-y)^2+2i(x+y)$ ；(7) $z \operatorname{Re} z$ ；(8) $1/z$ ；(9) $\cos z$ ；(10) $\operatorname{sh} z$ 。

由可导充分条件（25题）判别：

- (1) 全平面不可导，不解析；
(2) 全平面不可导，不解析；

(3) 全平面可导，解析， $(z^m)' = mz^{m-1}$ ；

(4) 全平面可导，解析， $(e^z)' = e^z$ ；

(5) 除 $(-1, -1)$ 点可导外，全平面其余处处不可导，全平面不解析；

(6) 除 $y=x-1$ 的线上处处可导外，其余点不可导，全平面不解析；

(7) $z=0$ 点可导， $(z \operatorname{Re} z)' \Big|_{z=0} = 0$ ，其余处处不可导，全平面不解析；

(8) 除 $z=0$ 点外在扩充全平面上可导，解析， $(1/z)' = -1/z^2$ ；

(9) 全平面可导，解析， $(\cos z)' = -\sin z$ ；

(10) 全平面可导，解析， $(\operatorname{sh} z)' = \operatorname{ch} z$ 。

5. 证明极坐标下的 Cauchy-Riemann 条件： $\frac{\partial u}{\partial \rho} = \frac{1}{\rho} \frac{\partial v}{\partial \varphi}$ ， $\frac{\partial v}{\partial \rho} = -\frac{1}{\rho} \frac{\partial u}{\partial \varphi}$ 。(1.11)

由变换关系 $x = \rho \cos \varphi$ ， $y = \rho \sin \varphi$ 可得

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cos \varphi + \frac{\partial u}{\partial y} \sin \varphi, \quad \frac{\partial u}{\partial \varphi} = -\rho \frac{\partial u}{\partial x} \sin \varphi + \rho \frac{\partial u}{\partial y} \cos \varphi,$$

$$\frac{\partial v}{\partial \rho} = \frac{\partial v}{\partial x} \cos \varphi + \frac{\partial v}{\partial y} \sin \varphi, \quad \frac{\partial v}{\partial \varphi} = -\rho \frac{\partial v}{\partial x} \sin \varphi + \rho \frac{\partial v}{\partial y} \cos \varphi.$$

$$\text{变换得 } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho} \cos \varphi - \frac{1}{\rho} \frac{\partial u}{\partial \varphi} \sin \varphi, \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho} \sin \varphi + \frac{1}{\rho} \frac{\partial u}{\partial \varphi} \cos \varphi, \quad (2)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \rho} \cos \varphi - \frac{1}{\rho} \frac{\partial v}{\partial \varphi} \sin \varphi, \quad (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \rho} \sin \varphi + \frac{1}{\rho} \frac{\partial v}{\partial \varphi} \cos \varphi. \quad (4)$$

$$\text{代入直角坐标的 C-R 方程 } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ 有}$$

$$\frac{\partial u}{\partial \rho} \cos \varphi - \frac{1}{\rho} \frac{\partial u}{\partial \varphi} \sin \varphi = \frac{\partial v}{\partial \rho} \sin \varphi + \frac{1}{\rho} \frac{\partial v}{\partial \varphi} \cos \varphi, \quad (5)$$

$$\frac{\partial u}{\partial \rho} \sin \varphi + \frac{1}{\rho} \frac{\partial u}{\partial \varphi} \cos \varphi = -\frac{\partial v}{\partial \rho} \cos \varphi + \frac{1}{\rho} \frac{\partial v}{\partial \varphi} \sin \varphi. \quad (6)$$

$$(5) \times \cos \varphi + (6) \times \sin \varphi \text{ 得 } \frac{\partial u}{\partial \rho} = \frac{1}{\rho} \frac{\partial v}{\partial \varphi},$$

$$(5) \times \sin \varphi - (6) \times \cos \varphi \text{ 得 } \frac{\partial v}{\partial \rho} = -\frac{1}{\rho} \frac{\partial u}{\partial \varphi}.$$

6. 设 $f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ 。(1) 证明：当 $z \rightarrow 0$ 时， $\frac{f(z)}{z}$ 的极限不存在；(2) 若

$$u = \operatorname{Re} f(z), \quad v = \operatorname{Im} f(z), \quad \text{证明：} u(x, 0) = x, \quad v(0, y) = y, \quad u(0, y) = v(x, 0) = 0;$$

(3) 证明： u ， v 的偏导数存在，且 C-R 方程成立，但 (1) 中已证明 $f'(0)$ 不存在，这个结论和 25 题矛盾吗？

$$(1) \quad z \neq 0 \text{ 时, } \frac{f(z)}{z} = \frac{z^4}{|z|^4} = \frac{(x^2-y^2)^2 - 4x^2y^2 + 4ixy(x^2-y^2)}{(x^2+y^2)^2}.$$

$$\frac{f(z)}{z} = \frac{(1-k^2)^2 - 4k^2 + 4ik(1-k^2)}{(1+k^2)^2}, \quad \text{可见 } z \text{ 沿不同直线趋于 } 0 \text{ 将有不同极限值, 所以}$$

$$\frac{f(z)}{z} \text{ 的极限不存在.}$$

$$(2) \quad z \neq 0 \text{ 时, } f(z) = \frac{x^5 - 10x^3y^2 + 5xy^4 + i(5x^4y - 10x^2y^3 + y^5)}{(x^2+y^2)^2}.$$

$$\text{所以 } u = \begin{cases} \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}, \quad v = \begin{cases} \frac{5x^4y - 10x^2y^3 + y^5}{(x^2+y^2)^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}.$$

$$\text{容易看出 } u(x, 0) = x, \quad v(0, y) = y, \quad u(0, y) = v(x, 0) = 0.$$

(3) 仿 (1) 的方法， u ， v 在 $z=0$ 处的偏导数不存在。

7. 利用极坐标下的 C-R 方程 (24 题) 证明: $f'(z) = \frac{\rho}{z} \left(\frac{\partial u}{\partial \rho} + i \frac{\partial v}{\partial \rho} \right) = \frac{1}{z} \left(\frac{\partial v}{\partial \varphi} - i \frac{\partial u}{\partial \varphi} \right)$ 。

利用 24 题 (1) (3) 式,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial \rho} \cos \varphi - \frac{1}{\rho} \frac{\partial u}{\partial \varphi} \sin \varphi + i \frac{\partial v}{\partial \rho} \cos \varphi - i \frac{1}{\rho} \frac{\partial v}{\partial \varphi} \sin \varphi$$

代入极坐标 C-R 方程,

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial \rho} \cos \varphi + \frac{\partial v}{\partial \rho} \sin \varphi + i \frac{\partial v}{\partial \rho} \cos \varphi - i \frac{\partial u}{\partial \rho} \sin \varphi \\ &= \frac{\rho}{z} (\cos \varphi + i \sin \varphi) \left(\frac{\partial u}{\partial \rho} \cos \varphi + \frac{\partial v}{\partial \rho} \sin \varphi + i \frac{\partial v}{\partial \rho} \cos \varphi - i \frac{\partial u}{\partial \rho} \sin \varphi \right) \\ &= \frac{\rho}{z} \left(\frac{\partial u}{\partial \rho} + i \frac{\partial v}{\partial \rho} \right) \end{aligned}$$

$$\text{再利用极坐标 C-R 方程有 } f'(z) = \frac{\rho}{z} \left(\frac{\partial u}{\partial \rho} + i \frac{\partial v}{\partial \rho} \right) = \frac{1}{z} \left(\frac{\partial v}{\partial \varphi} - i \frac{\partial u}{\partial \varphi} \right)。$$

8. 设 $\rho = \rho(x, y)$, $\varphi = \varphi(x, y)$ 是实变量 x, y 的实函数。若 $f(z) = \rho(\cos \varphi + i \sin \varphi)$ 是

$$z = x + iy \text{ 的解析函数, 证明: } \frac{\partial \rho}{\partial x} = \rho \frac{\partial \varphi}{\partial y}, \quad \frac{\partial \rho}{\partial y} = -\rho \frac{\partial \varphi}{\partial x}。$$

$$\frac{\partial u}{\partial x} = \frac{\partial \rho}{\partial x} \cos \varphi - \rho \sin \varphi \frac{\partial \varphi}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial \rho}{\partial y} \cos \varphi - \rho \sin \varphi \frac{\partial \varphi}{\partial y},$$

$$\frac{\partial v}{\partial x} = \frac{\partial \rho}{\partial x} \sin \varphi + \rho \cos \varphi \frac{\partial \varphi}{\partial x}, \quad \frac{\partial v}{\partial y} = \frac{\partial \rho}{\partial y} \sin \varphi + \rho \cos \varphi \frac{\partial \varphi}{\partial y}。$$

由 C-R 方程可得:

$$\frac{\partial \rho}{\partial x} \cos \varphi - \rho \sin \varphi \frac{\partial \varphi}{\partial x} = \frac{\partial \rho}{\partial y} \sin \varphi + \rho \cos \varphi \frac{\partial \varphi}{\partial y}, \quad (1)$$

$$\frac{\partial \rho}{\partial x} \sin \varphi + \rho \cos \varphi \frac{\partial \varphi}{\partial x} = -\frac{\partial \rho}{\partial y} \cos \varphi + \rho \sin \varphi \frac{\partial \varphi}{\partial y}。 \quad (2)$$

$$(1) \times \cos \varphi + (2) \times \sin \varphi \text{ 得 } \frac{\partial \rho}{\partial x} = \rho \frac{\partial \varphi}{\partial y},$$

$$(1) \times \sin \varphi - (2) \times \cos \varphi \text{ 得 } \frac{\partial \rho}{\partial y} = -\rho \frac{\partial \varphi}{\partial x}。$$

9. 设 $r = r(\rho, \varphi)$, $\theta = \theta(\rho, \varphi)$ 是实变量 ρ, φ 的实函数。若 $f(z) = r(\cos \theta + i \sin \theta)$ 解

$$\text{析, 其中 } z = \rho e^{i\varphi}, \text{ 试证: } \frac{\partial r}{\partial \rho} = \frac{r}{\rho} \frac{\partial \theta}{\partial \varphi}, \quad \frac{\partial r}{\partial \varphi} = -\rho r \frac{\partial \theta}{\partial \rho}。$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial r}{\partial \rho} \cos \theta - r \sin \theta \frac{\partial \theta}{\partial \rho}, \quad \frac{\partial u}{\partial \varphi} = \frac{\partial r}{\partial \varphi} \cos \theta - r \sin \theta \frac{\partial \theta}{\partial \varphi},$$

$$\frac{\partial v}{\partial \rho} = \frac{\partial r}{\partial \rho} \sin \theta + r \cos \theta \frac{\partial \theta}{\partial \rho}, \quad \frac{\partial v}{\partial \varphi} = \frac{\partial r}{\partial \varphi} \sin \theta + r \cos \theta \frac{\partial \theta}{\partial \varphi}。$$

由极坐标 C-R 方程 (24 题) 得:

$$\frac{\partial r}{\partial \rho} \cos \theta - r \sin \theta \frac{\partial \theta}{\partial \rho} = \frac{1}{\rho} \frac{\partial r}{\partial \varphi} \sin \theta + \frac{r}{\rho} \cos \theta \frac{\partial \theta}{\partial \varphi}, \quad (1)$$

$$\frac{\partial r}{\partial \rho} \sin \theta + r \cos \theta \frac{\partial \theta}{\partial \rho} = -\frac{1}{\rho} \frac{\partial r}{\partial \varphi} \cos \theta + \frac{r}{\rho} \sin \theta \frac{\partial \theta}{\partial \varphi}。 \quad (2)$$

$$(1) \times \cos \varphi + (2) \times \sin \varphi \text{ 得 } \frac{\partial r}{\partial \rho} = \frac{r}{\rho} \frac{\partial \theta}{\partial \varphi},$$

$$(2) \times \sin \varphi - (1) \times \cos \varphi \text{ 得 } \frac{\partial r}{\partial \varphi} = -\rho r \frac{\partial \theta}{\partial \rho}。$$

三. 初等函数与多值函数

1. 证明下列不等式: (1) $|\operatorname{sh} y| \leq |\sin(x + iy)| \leq \operatorname{ch} y;$ (2.2)

$$(2) |\operatorname{sh} y| \leq |\cos(x + iy)| \leq \operatorname{ch} y。$$

$$(1) \sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \operatorname{ch} y + i \cos x \operatorname{sh} y,$$

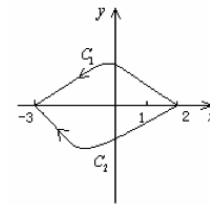
$$\text{所以 } |\sin(x + iy)| = \sqrt{\sin^2 x \operatorname{ch}^2 y + \cos^2 x \operatorname{sh}^2 y}。$$

$$\text{代入 } \operatorname{ch}^2 y = 1 + \operatorname{sh}^2 y \text{ 得 } |\sin(x + iy)| = \sqrt{\sin^2 x + \operatorname{sh}^2 y} \geq \operatorname{sh} y,$$

$$\text{代入 } \operatorname{sh}^2 y = \operatorname{ch}^2 y - 1 \text{ 得 } |\sin(x + iy)| = \sqrt{\operatorname{ch}^2 y - \cos^2 x} \leq \operatorname{ch} y。 \text{ 不等式得证。}$$

$$(2) \text{ 同 } (1)。$$

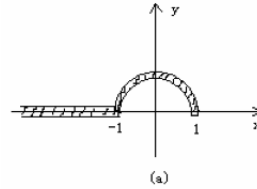
11. 函数 $w = z + \sqrt{z-1}$, 规定 $w(2) = 1$, 是分别求当 z 沿着图中的 C_1 和 C_2 连续变化时 $w(-3)$ 之值。 (2.5)



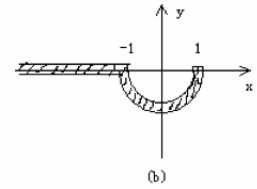
若规定 $z = 2$ 处 $\arg(z-1) = 2\pi$, 则有 $w(2) = 1$ 。 z 沿 C_1 连续变化到 -3 时, $\arg(z-1) = 3\pi$,

所以 $w(-3) = -3 + \sqrt{4e^{i\frac{3\pi}{2}}} = -3 - 2i$ 。沿 C_2 有 $w(-3) = -3 + 2i$ 。

12. 函数 $w = \ln(1 - z^2)$, 规定 $w(0) = 0$, 试讨论当 z 分别限制在以下两图中变化时, $w(3)$ 之值。



(a)



(b)

(a) $w = \ln[(1-z)(z+1)]$ 。规定 $z = 0$ 时 $\arg(1-z) = 0$, $\arg(z+1) = 0$ 就有 $w(0) = 0$ 。

z 从下半平面到达 $z = 3$ 时有 $\arg(1-z) = \pi$, $\arg(z+1) = 0$, 所以

$$w(3) = \ln(2e^{i\pi} \cdot 4e^{i0}) = 3 \ln 2 + i\pi。$$

(b) z 从上半平面到达 $z = 3$ 时有 $\arg(1-z) = -\pi$, $\arg(z+1) = 0$, 所以

$$w(3) = \ln(2e^{-i\pi} \cdot 4e^{i0}) = 3 \ln 2 - i\pi。$$

13. $f(z) = \frac{z^{1-p}(1-z)^p}{2z}$, $-1 < p < 2$ 。在实轴上沿 0 到 1 做割线, 规定沿割线上岸 $\arg z = \arg(1-z) = 0$, 试计算 $f(\pm i)$ 。 (2.8)

$$z = i \text{ 时, } \arg z = \frac{\pi}{2}, \arg(1-z) = -\frac{\pi}{4}, f(i) = \frac{(e^{i\pi/2})^{1-p} (2^{1/2} e^{-i\pi/4})^p}{2i} = 2^{\frac{p}{2}-1} e^{\frac{3}{4}p\pi}。$$

$$z \text{ 从左边由割线上岸绕到 } z = -i, \text{ 则 } \arg z = \frac{3\pi}{2}, \arg(1-z) = \frac{\pi}{4}, f(-i) = 2^{\frac{p}{2}-1} e^{-i\frac{5}{4}p\pi}。$$

$$z \text{ 从右边由割线上岸绕到 } z = -i, \text{ 则 } \arg z = -\frac{\pi}{2}, \arg(1-z) = \frac{7\pi}{4}, f(-i) = 2^{\frac{p}{2}-1} e^{-i\frac{5}{4}p\pi}。$$

四复变积分

14. 计算: (1) $\int_{|z|=1} \frac{dz}{z}$; (2) $\int_{|z|=1} \frac{dz}{|z|}$; (3) $\int_{|z|=1} \frac{|dz|}{z}$; (4) $\int_{|z|=1} \left| \frac{dz}{z} \right|$ 。

(1) 令 $z = e^{i\varphi}$, $\int_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} \frac{de^{i\varphi}}{e^{i\varphi}} = \int_0^{2\pi} id\varphi = 2\pi i$; (3.2)

(2) $\int_{|z|=1} \frac{dz}{|z|} = \int_0^{2\pi} de^{i\varphi} = e^{i\varphi} \Big|_0^{2\pi} = 0$;

(3) $\int_{|z|=1} \frac{|dz|}{z} = \int_0^{2\pi} \frac{|ie^{i\varphi} d\varphi|}{e^{i\varphi}} = \int_0^{2\pi} e^{-i\varphi} d\varphi = ie^{-i\varphi} \Big|_0^{2\pi} = 0$;

(4) $\int_{|z|=1} \left| \frac{dz}{z} \right| = \int_0^{2\pi} d\varphi = 2\pi$ 。

15. 计算: (1) $\oint_{|z|=2} \frac{\cos z}{z} dz$; (2) $\oint_{|z|=2} \frac{\sin z}{z^2} dz$; (3) $\oint_{|z|=2} \frac{z^2}{z-1} dz$; (4) $\oint_{|z|=2} \frac{z^2-1}{z^2+1} dz$;

(5) $\oint_{|z|=2} \frac{dz}{z^2}$; (6) $\oint_{|z|=2} \frac{dz}{z^2+z+1}$; (7) $\oint_{|z|=2} \frac{dz}{z^2-8}$; (8) $\oint_{|z|=2} \frac{dz}{z^2-2z+3}$;

(9) $\oint_{|z|=2} \frac{|z|e^z dz}{z^2}$; (10) $\oint_{|z|=2} \frac{dz}{z^2(z^2+16)}$ 。

(1) $\oint_{|z|=2} \frac{\cos z}{z} dz = 2\pi i \cdot \cos z \Big|_{z=0} = 2\pi i$;

(2) $\oint_{|z|=2} \frac{\sin z}{z^2} dz = 2\pi i (\sin z)' \Big|_{z=0} = 2\pi i$;

(3) $\oint_{|z|=2} \frac{z^2}{z-1} dz = 2\pi i \cdot z^2 \Big|_{z=1} = 2\pi i$;

(4) $\oint_{|z|=2} \frac{z^2-1}{z^2+1} dz = 2\pi i \left(\frac{z^2-1}{z+i} \Big|_{z=i} + \frac{z^2-1}{z-i} \Big|_{z=-i} \right) = 0$;

(5) $\oint_{|z|=2} \frac{dz}{z^2} = 0$;

(6) $\oint_{|z|=2} \frac{dz}{z^2+z+1} = 2\pi i \left(\frac{1}{z+1/2-i\sqrt{3}/2} \Big|_{z=-1/2-i\sqrt{3}/2} + \frac{1}{z+1/2+i\sqrt{3}/2} \Big|_{z=-1/2+i\sqrt{3}/2} \right) = 0$;

(7) $\oint_{|z|=2} \frac{dz}{z^2-8} = 0$; (不包围奇点)

(8) $\oint_{|z|=2} \frac{dz}{z^2-2z+3} = 2\pi i \left(\frac{1}{z-1-i\sqrt{2}} \Big|_{z=1-i\sqrt{2}} + \frac{1}{z-1+i\sqrt{2}} \Big|_{z=1+i\sqrt{2}} \right) = 0$;

(9) $\oint_{|z|=2} \frac{|z|e^z dz}{z^2} = 2 \oint_{|z|=2} \frac{e^z dz}{z^2} = 4\pi i (e^z)' \Big|_{z=0}$;

(10) $\oint_{|z|=2} \frac{dz}{z^2(z^2+16)} = 2\pi i \left(\frac{1}{z^2+16} \right)' \Big|_{z=0} = 0$ 。

16. (1) 计算 $\oint_{|z|=1} \frac{e^z}{z^3} dz$; (2) 对于什么样的 a 值, 函数 $F(z) = \int_{z_0}^z e^t \left(\frac{1}{t} + \frac{a}{t^3} \right) dt$ 是单值的?

(1) $\oint_{|z|=1} \frac{e^z}{z^3} dz = \frac{2\pi i}{2!} (e^z)'' \Big|_{z=0} = \pi i$;

(2) $\oint_C e^t \left(\frac{1}{t} + \frac{a}{t^3} \right) dt = \begin{cases} 0, & C \text{ 不包围原点} \\ (2+a)\pi i, & C \text{ 包围原点} \end{cases}$ 。当 $a = -2$ 时, 对任意的闭曲线 (不过原点) 该积分都是 0, 则 $F(z)$ 为单值函数。

17. 设 G 是单连通区域, C 是它的边界, $z_1, z_2, \dots, z_n$ 是 G 内的 n 个不同的点。

$P(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$, $f(z)$ 在 G 中解析, 证明: $Q(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) P(\zeta) - P(z)}{P(\zeta)(\zeta-z)} d\zeta$

是一个 $n-1$ 次多项式, 且 $Q(z_k) = f(z_k)$, $k=1, 2, \dots, n$ 。如果 G 是复连通区域, 上述结果还正确吗?

证: $z \neq z_k$ 时, $Q(z) = \sum_{i=1}^n \left[(\zeta-z_i) \frac{f(\zeta) P(\zeta) - P(z)}{P(\zeta)(\zeta-z)} \right]_{\zeta=z_i}$

$$= \sum_{i=1}^n \left[\frac{f(z_i)(\zeta-z_i)}{\prod_{j=1}^n (\zeta-z_j)} \right]_{\zeta=z_i} \cdot \frac{\prod_{j=1}^n (z-z_j)}{z-z_i} = \sum_{i=1}^n \left[\frac{f(z_i)}{\prod_{j \neq i} (z_i-z_j)} \prod_{j \neq i} (z-z_j) \right]$$

即它是 $n-1$ 次多项式。

$z = z_k$ 时, $Q(z_k) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta-z_k} d\zeta = f(z_k)$ 。

若 G 是复连通区域, 上面的计算不成立。

18. 设 $f(z)$ 在 $|z| \leq R$ 的区域内解析, 且 $\zeta = \rho e^{i\varphi}$ ($0 \leq \rho < R$) 为圆内一点, 证明圆内的

Poisson 公式: $f(\zeta) = \frac{R^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} d\theta$ 。 (3.8)

证: $f(\zeta) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z-\zeta} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{i\theta})}{Re^{i\theta} - \rho e^{i\varphi}} dRe^{i\theta} = \frac{1}{2\pi} \int_0^{2\pi} \frac{Rf(Re^{i\theta})}{R - \rho e^{i(\varphi-\theta)}} d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{R \left[\frac{R - \rho e^{-i(\varphi-\theta)}}{R - \rho e^{i(\varphi-\theta)}} \right]}{\left[R - \rho e^{i(\varphi-\theta)} \right] \left[R - \rho e^{-i(\varphi-\theta)} \right]} f(Re^{i\theta}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - R\rho e^{-i(\varphi-\theta)}}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} f(Re^{i\theta}) d\theta \quad (1)$$

令 $\zeta' = \frac{R^2}{\rho} e^{i\varphi}$, 它在圆外, 所以有 $\frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z-\zeta'} dz = 0$ (函数 $\frac{f(z)}{z-\zeta'}$ 在圆内解析)。

$0 = \oint_{|z|=R} \frac{f(z)}{z-\zeta'} dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - R\rho e^{-i(\varphi-\theta)}}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} f(Re^{i\theta}) d\theta \quad (2)$

(1) - (2) 即得 $f(\zeta) = \frac{R^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} d\theta$ 。

19. 设 $f(z)$ 在区域 G 内解析, C 为 G 内任一简单闭曲线, 证明对于 G 内, 但不在 C 上的

任一点 z , $\oint_C \frac{f'(\zeta)}{\zeta-z} d\zeta = \oint_C \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$ 。

由 Cauchy 积分公式, $\oint_C \frac{f'(\zeta)}{\zeta-z} d\zeta = 2\pi i f'(z)$ 。由解析函数高阶导数公式,

$\oint_C \frac{f(\zeta)}{(\zeta-z)^2} d\zeta = 2\pi i f'(z)$, 得证。

20. 函数 $f(z)$ 在 G 内解析, 且 z_0 为 G 内一点, 有 $f'(z_0) \neq 0$, 试证明:

$$\frac{2\pi i}{f'(z_0)} = \oint_C \frac{dz}{f(z) - f(z_0)}.$$

其中 C 是以 z_0 为圆心的一个足够小的圆。

证: 令 $F(z) = \begin{cases} \frac{z - z_0}{f(z) - f(z_0)}, & z \neq z_0 \\ \frac{1}{f'(z_0)}, & z = z_0 \end{cases}$, $z \neq z_0$ 时 $F(z)$ 显然是可导的, 对于 $z = z_0$ 点,

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{F(z) - F(z_0)}{z - z_0} &= \lim_{z \rightarrow z_0} \frac{(z - z_0)f'(z_0) - f(z) + f(z_0)}{f'(z_0)(z - z_0)[f(z) - f(z_0)]} \\ &= \lim_{z \rightarrow z_0} \frac{f'(z_0) - f'(z)}{f'(z_0)[f(z) - f(z_0)] + f'(z_0)f'(z)(z - z_0)} \\ &= \lim_{z \rightarrow z_0} \frac{-\frac{f'(z) - f'(z_0)}{z - z_0}}{f'(z_0)\frac{f(z) - f(z_0)}{z - z_0} + f'(z_0)f'(z)} = -\frac{f''(z_0)}{2[f'(z_0)]^2}. \end{aligned}$$

即 $F(z)$ 在 $z = z_0$ 点也是可导的, 所以 $F(z)$ 在 G 内解析, 因此有:

$$\frac{1}{2\pi i} \oint_C \frac{dz}{f(z) - f(z_0)} = \frac{1}{2\pi i} \oint_C \frac{F(z)}{z - z_0} dz = F(z_0) = \frac{1}{f'(z_0)}.$$

21. 函数 $f(z)$, $g(z)$ 及 $g(z)$ 的反函数均在 G 内单值解析, 且 $g'(z)$ 恒不为 0, 试计算

$$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{g(\zeta) - g(z)} d\zeta,$$

其中 C 是 G 内的简单闭曲线, z 不在 C 上。

由于 $g(z)$ 的反函数在 G 内单值, 所以当且仅当 $\zeta = z$ 时 $g(\zeta) = g(z)$, 即 $\frac{f(\zeta)}{g(\zeta) - g(z)}$ 在

G 内只有一个奇点 $\zeta = z$ 。

若 C 不包围 z , 则 $\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{g(\zeta) - g(z)} d\zeta = 0$ 。

若 C 包围 z , 同上题作法, 令 $F(\zeta) = \begin{cases} \frac{\zeta - z}{g(\zeta) - g(z)} f(\zeta), & \zeta \neq z \\ \frac{f(z)}{g'(z)}, & \zeta = z \end{cases}$, 它在 G 内解析, 所以

有: $\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{g(\zeta) - g(z)} d\zeta = \frac{1}{2\pi i} \oint_C \frac{F(\zeta)}{\zeta - z} d\zeta = F(z) = \frac{f(z)}{g'(z)}.$

五. 无穷级数

22. 设 $\sum a_n$ 与 $\sum b_n$ 皆为正项级数, 试举反例, 说明下列说法不对: (4.1)

(1) 若 $\lim_{n \rightarrow \infty} na_n = 0$, 则 $\sum a_n$ 收敛; (2) 若 $a_{2n} < a_{2n+1}$, 则 $\sum a_n$ 发散;

(3) 若 $\lim_{n \rightarrow \infty} \frac{a_{2n+1}}{a_n} = \infty$, 则 $\sum a_n$ 发散; (4) 若 $\sum a_n$ 与 $\sum b_n$ 发散, 则 $\sum \sqrt{a_n b_n}$ 发散。

(1) 取 $a_n = \frac{1}{n \ln n}$; (可用积分判别法断定级数 $\sum \frac{1}{n(\ln n)^p}$ 当 $p > 1$ 时收敛, $p \leq 1$ 时发散)

(2) 取 $a_n = \frac{2 - (-1)^n}{n^2}$, 则 $n \geq 1$ 时 $a_{2n} - a_{2n+1} = \frac{-8n^2 + 4n + 1}{(2n)^2(2n+1)^2} < 0$, 而 $a_n \leq \frac{3}{n^3}$, $\sum \frac{3}{n^3}$

收敛, 所以 $\sum a_n$ 收敛;

(3) 取 $a_n = n^{-2 \frac{(-1)^n}{2}}$, 则 $\lim_{n \rightarrow \infty} \frac{a_{2n+1}}{a_n} = \lim_{n \rightarrow \infty} n = \infty$, 而 $a_n \leq n^{-\frac{3}{2}}$, $\sum \frac{1}{n^{3/2}}$ 是收敛的, 所以 $\sum a_n$

收敛;

(4) 取 $a_n = \frac{1}{n^{2+(-1)^n}}$, $b_n = \frac{1}{n^{2-(-1)^n}}$, 因为 $\sum_n a_n = \sum_k \frac{1}{(2k)^3} + \sum_k \frac{1}{2k+1}$, 右边第一个级数

收敛, 第二个发散, 所以 $\sum a_n$ 发散, 同样的, $\sum b_n$ 也发散, 而 $\sum \sqrt{a_n b_n} = \sum \frac{1}{n^2}$ 收敛。

23. 判断下列级数的收敛性及绝对收敛性: (1) $\sum \frac{i^n}{\ln n}$; (2) $\sum \frac{i^n}{n}$ 。 (4.2)

(1) $\sum_n \frac{i^n}{\ln n} = \sum_k \frac{(-1)^k}{\ln(2k)} + i \sum_k \frac{(-1)^k}{\ln(2k+1)}$, 右边两个级数都收敛 (用 Leibnitz 判别法),

所以 $\sum \frac{i^n}{\ln n}$ 收敛, 因为 $\left| \frac{i^n}{\ln n} \right| = \frac{1}{\ln n} > \frac{1}{n}$, 而 $\sum \frac{1}{n}$ 发散, 所以 $\sum \frac{i^n}{\ln n}$ 不绝对收敛

(2) 同上, $\sum \frac{i^n}{n} = \sum \frac{(-1)^k}{2k} + i \sum \frac{(-1)^k}{2k+1}$, 所以 $\sum \frac{i^n}{n}$ 收敛。 $\left| \frac{i^n}{n} \right| = \frac{1}{n}$, $\sum \frac{1}{n}$ 发散, 所以

$\sum \frac{i^n}{n}$ 不绝对收敛。

24. 证明级数 $\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$, $|z| \neq 1$ 收敛, 并求其和。 (4.3)

$$\begin{aligned} S_N(z) &= \sum_{n=1}^N \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{z(1-z)} \sum_{n=1}^N \left(\frac{1}{1-z^n} - \frac{1}{1-z^{n+1}} \right) \\ &= \frac{1}{z(1-z)} \left(\frac{1}{1-z} - \frac{1}{1-z^2} + \frac{1}{1-z^2} - \frac{1}{1-z^3} + \cdots + \frac{1}{1-z^N} - \frac{1}{1-z^{N+1}} \right) \\ &= \frac{1}{z(1-z)} \left(\frac{1}{1-z} - \frac{1}{1-z^{N+1}} \right) \end{aligned}$$

若 $|z| < 1$, $N \rightarrow \infty$ 时 $S_N(z) \rightarrow \frac{1}{(1-z)^2}$, 若 $|z| > 1$, $S_N(z) \rightarrow \frac{1}{z(1-z)^2}$, 即

$$S(z) = \lim_{N \rightarrow \infty} S_N(z) = \begin{cases} \frac{1}{(1-z)^2}, & |z| < 1 \\ \frac{1}{z(1-z)^2}, & |z| > 1 \end{cases}.$$

25. 证明级数 $\sum e^{-n} \sin nz$ 在区域 $|\operatorname{Im} z| < 1$ 内解析。 (4.4)

证: 对于任意 $p \in (0, 1)$, 当 $|\operatorname{Im} z| \leq p$ 时, (参考习题 02 的 43 题)

$$|e^{-n} \sin nz| \leq e^{-n} \operatorname{ch} ny \leq e^{-n} \operatorname{ch} pn = \frac{1}{2} [e^{-(1-p)n} + e^{-(1+p)n}].$$

因为级数 $\sum [e^{-(1-p)n} + e^{-(1+p)n}]$

收敛, 所以级数 $\sum e^{-n} \sin nz$ 在区域 $|\operatorname{Im} z| \leq p$ 内一致收敛, 由 Weierstrass 定理, 级数

$\sum e^{-n} \sin nz$ 在区域 $|\operatorname{Im} z| < p$ 内解析。这里的 p 具有任意性。

任取区域 $|\operatorname{Im} z| < 1$ 内一点 z_0 , 存在 p_0 使 $|\operatorname{Im} z_0| < p_0 < 1$ 。由于级数 $\sum e^{-n} \sin nz$ 在区域

$|\operatorname{Im} z| < p_0$ 内解析, 故在 z_0 点解析。由 z_0 的任意性, $\sum e^{-n} \sin nz$ 在区域 $|\operatorname{Im} z| < 1$ 内解析。

26. 已知幂级数 $\sum a_n z^n$ 和 $\sum b_n z^n$ 的收敛半径分别为 R_1, R_2 , 试讨论下列幂级数的收敛半

径: (1) $\sum (a_n + b_n) z^n$; (2) $\sum a_n b_n z^n$; (3) $\sum \frac{1}{a_n} z^n$; (4) $\sum \frac{b_n}{a_n} z^n$ 。 (4.6)

(1) 设 $R_1 < R_2$ 。 $|z| < R_1$ 时, 级数收敛, $R_1 < |z| < R_2$ 时, 级数发散; 由阿贝尔第一定理,

当 $|z| > R_2$ 时级数也发散, 所以 $R = R_1$ 。

若 $R_1 = R_2$, $|z| < R_1 = R_2$ 时, 级数收敛, $|z| > R_1 = R_2$ 时, 级数有可能收敛。

综上, $R \geq \min\{R_1, R_2\}$ 。

(2) 参考数学分析中关于上下极限的等式和不等式。

$$R = \lim_{n \rightarrow \infty} \frac{1}{|a_n b_n|^{1/n}} \geq \lim_{n \rightarrow \infty} \frac{1}{|a_n|^{1/n}} \cdot \lim_{n \rightarrow \infty} \frac{1}{|b_n|^{1/n}} = R_1 R_2;$$

$$(3) R = \frac{1}{\lim_{n \rightarrow \infty} \frac{1}{|a_n|^{1/n}}} \leq \frac{1}{\lim_{n \rightarrow \infty} \frac{1}{|a_n|^{1/n}}} = \frac{1}{R_1};$$

$$(4) R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{b_n} \right|^{1/n} \leq \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} \cdot \lim_{n \rightarrow \infty} \frac{1}{|b_n|^{1/n}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{1}{|a_n|^{1/n}}} \cdot \lim_{n \rightarrow \infty} \frac{1}{|b_n|^{1/n}} = \frac{R_2}{R_1}.$$

27. 确定下列级数的收敛半径（或收敛区域）：（1） $\sum \frac{1}{n^n} z^n$ ；（2） $\sum \frac{1}{2^n n^n} z^n$ ；（3） $\sum \frac{n!}{n^n} z^n$ ；（4） $\sum \frac{(-1)^n}{2^{2n} (n!)^2} z^n$ ；（5） $\sum n^{\ln n} z^n$ ；（6） $\sum z^n$ ；（7） $\sum \frac{1}{2^{2n}} z^{2n}$ ；

$$(8) \sum \left(\frac{z}{1+z} \right)^n; (9) \sum (-1)^n (z^2 + 2z + 2)^n; (10) \sum 2^n \sin \frac{z}{3^n}; (11) \sum \frac{\ln(n)}{n!} z^n;$$

$$(12) \sum \left(1 - \frac{1}{n} \right)^n z^n.$$

$$(1) \lim_{n \rightarrow \infty} \left| n^n \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} n = \infty, \text{ 所以收敛半径 } R = \infty;$$

$$(2) \lim_{n \rightarrow \infty} \left| 2^n n^n \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 2n = \infty, \text{ 所以收敛半径 } R = \infty;$$

$$(3) \lim_{n \rightarrow \infty} \left| \frac{n!}{n^n} \right| / \left| \frac{(n+1)!}{(n+1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e, \text{ 所以收敛半径 } R = e;$$

$$(4) \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{2^{2n} (n!)^2} \right| / \left| \frac{(-1)^{n+1}}{2^{2(n+1)} [(n+1)!]^2} \right| = \lim_{n \rightarrow \infty} 4(n+1)^2 = \infty, \text{ 所以收敛半径 } R = \infty;$$

$$(5) \lim_{n \rightarrow \infty} \left| n^{\ln n} \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} n^{\frac{\ln n}{n}} = \lim_{n \rightarrow \infty} e^{\frac{(\ln n)^2}{n}} = 1, \text{ 所以收敛半径 } R = 1;$$

$$(6) \text{ 显然收敛半径 } R = 1;$$

$$(7) \frac{1}{2^{2n}} z^{2n} = \left(\frac{z}{2} \right)^{2n}, \text{ 收敛半径 } R = 2;$$

$$(8) \text{ 收敛域为 } \left| \frac{z}{1+z} \right| < 1, \text{ 化简得 } \operatorname{Re} z > -\frac{1}{2};$$

$$(9) \text{ 收敛域为 } |z^2 + 2z + 2| < 1;$$

$$(10) \sum 2^n \sin \frac{z}{3^n} \text{ 在全平面收敛, 即收敛域为全平面};$$

$$(11) \lim_{n \rightarrow \infty} \left| \frac{n \ln n}{n!} \right| / \left| \frac{(n+1) \ln(n+1)}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \frac{n \ln n}{\ln(n+1)} = \infty, \text{ 所以收敛半径 } R = \infty;$$

$$(12) \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{n} \right)^n \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = 1, \text{ 所以收敛半径 } R = 1.$$

六. Taylor 与 Laurent 展开

28. 将下列函数在指定点展成 Taylor 级数，并给出其收敛半径：

$$(1) \sin z, \text{ 在 } z = n\pi \text{ 展开}; (2) 1 - z^2, \text{ 在 } z = 1 \text{ 展开}; (3) \frac{1}{1+z+z^2}, \text{ 在 } z = 0 \text{ 展开};$$

$$(4) \ln z, \text{ 在 } z = i \text{ 展开, 规定: (i) } 0 \leq \arg z < 2\pi, \text{ (ii) } -\pi \leq \arg z < \pi, \text{ (iii) } (\ln z)_{z=i} = -\frac{3}{2}\pi i;$$

$$(5) \arctan z \text{ 的主值, 在 } z = 0 \text{ 展开}; (6) \frac{\sin z}{1-z}, \text{ 在 } z = 0 \text{ 展开};$$

$$(7) \exp\left(\frac{1}{1-z}\right), \text{ 在 } z = 0 \text{ 展开 (可只求前四项系数)}; (8) \ln\left(\frac{1+z}{1-z}\right), \text{ 在 } z = \infty \text{ 展开}.$$

$$(1) (\sin z)_{z=n\pi}^{(k)} = \sin\left(n\pi + \frac{k}{2}\pi\right) = \begin{cases} 0, k=2m \\ (-1)^{n+m}, k=2m+1 \end{cases}$$

$$\sin z = \sum_{k=0}^{\infty} \frac{1}{k!} (\sin z)_{z=n\pi}^{(k)} (z-n\pi)^k = \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{(2m+1)!} (z-n\pi)^{2m+1};$$

$$(2) 1 - z^2 = -(z-1)(z+1) = -(z-1)[2 + (z-1)] = -2(z-1) - (z-1)^2;$$

$$(3) \frac{1}{1+z+z^2} = \frac{1}{(z-e^{i2\pi/3})(z-e^{-i2\pi/3})} = \frac{1}{\sqrt{3}i} \left(\frac{1}{z-e^{i2\pi/3}} - \frac{1}{z-e^{-i2\pi/3}} \right) \\ = \frac{1}{\sqrt{3}i} \left(\frac{e^{i2\pi/3}}{1-e^{i2\pi/3}z} - \frac{e^{-i2\pi/3}}{1-e^{-i2\pi/3}z} \right) \\ = \frac{1}{\sqrt{3}i} \sum_{n=0}^{\infty} \left[e^{i2\pi/3} (e^{i2\pi/3}z)^n - e^{-i2\pi/3} (e^{-i2\pi/3}z)^n \right] \\ = \frac{1}{\sqrt{3}i} \sum_{n=0}^{\infty} \left[e^{i2(n+1)\pi/3} - e^{-i2(n+1)\pi/3} \right] z^n = \frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \sin\left[\frac{2}{3}(n+1)\pi\right] z^n \quad (|z| < 1);$$

$$(4) (\ln z)_{z=i}^{(k)} = \left(\frac{1}{z} \right)_{z=i}^{(k-1)} = \frac{(-1)^{k-1} (k-1)!}{z^k} \bigg|_{z=i} = -\frac{(-1)^k (k-1)!}{i^k} = -\frac{i^{2k} (k-1)!}{i^k} = -i^k (k-1)!,$$

$$(k=1, 2, 3, \dots). \quad \ln z = (\ln z)_{z=i} + \sum_{k=1}^{\infty} \frac{1}{k!} (\ln z)_{z=i}^{(k)} (z-i)^k = (\ln z)_{z=i} - \sum_{k=1}^{\infty} \frac{i^k}{k} (z-i)^k,$$

$$(|z-i| < 1); \text{ (ii) } (\ln z)_{z=i} = \frac{1}{2}\pi i; \text{ (iii) } (\ln z)_{z=i} = -\frac{3}{2}\pi i \text{ 代入上式即可}.$$

$$(5) \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \quad (|t| < 1)$$

对两边积分，由于幂级数在收敛域内可逐项积分，故有

$$\arctan z = \int_0^z \frac{1}{1+t^2} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^z t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}, \quad (|z| < 1)$$

$$(6) \left(\frac{\sin z}{1-z} \right)_{z=0}^{(n)} = \sum_{k=0}^n \frac{n!}{k! (n-k)!} \left(\frac{1}{1-z} \right)_{z=0}^{(n-k)} (\sin z)_{z=0}^{(k)} = \sum_{k=0}^n \frac{n!}{k!} \sin \frac{k}{2}\pi = \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^m n!}{(2m+1)!}$$

$$\frac{\sin z}{1-z} = \left(\frac{\sin z}{1-z} \right)_{z=0} + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\sin z}{1-z} \right)_{z=0}^{(n)} z^n = \sum_{n=1}^{\infty} \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^m}{(2m+1)!} z^n, \quad (|z| < 1)$$

$$(7) \text{ 记 } f(z) = e^{\frac{1}{1-z}}, \text{ 则 } f(0) = e. \quad f'(z) = \frac{1}{(1-z)^2} e^{\frac{1}{1-z}}, \quad f'(0) = e.$$

$$f''(z) = \left[\frac{2}{(1-z)^3} + \frac{1}{(1-z)^4} \right] e^{\frac{1}{1-z}}, \quad f''(0) = 3e.$$

$$f'''(z) = \left[\frac{6}{(1-z)^4} + \frac{6}{(1-z)^5} + \frac{1}{(1-z)^6} \right] e^{\frac{1}{1-z}}, \quad f'''(0) = 13e.$$

$$f^{(4)}(z) = \left[\frac{24}{(1-z)^5} + \frac{36}{(1-z)^6} + \frac{12}{(1-z)^7} + \frac{1}{(1-z)^8} \right] e^{\frac{1}{1-z}}, \quad f^{(4)}(0) = 73e.$$

$$f(z) = f(0) + f'(0)z + \frac{1}{2}f''(0)z^2 + \frac{1}{6}f'''(0)z^3 + \frac{1}{24}f^{(4)}(0)z^4 + \dots \\ = e \left(1 + z + \frac{3}{2}z^2 + \frac{13}{6}z^3 + \frac{73}{24}z^4 + \dots \right) \quad (|z| < 1)$$

$$(8) \text{ 记 } f(z) = \ln \frac{1+z}{1-z}, \text{ 则 } f\left(\frac{1}{t}\right) = \ln \frac{t+1}{t-1} = \ln(t+1) - \ln(t-1).$$

$$\ln(t+1) = \ln(t+1)_{t=0} + \int_0^t \frac{1}{1+u} du = \ln(t+1)_{t=0} + \sum_{n=0}^{\infty} (-1)^n u^n du \\ = \ln(t+1)_{t=0} + \sum_{n=0}^{\infty} (-1)^n \int_0^t u^n du = \ln(t+1)_{t=0} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1}, \quad (|t| < 1)$$

$$-\ln(t-1) = -\ln(t-1)_{t=0} + \int_0^t \frac{1}{1-u} du = -\ln(t-1)_{t=0} + \sum_{n=0}^{\infty} \int_0^t u^n du$$

$$= -\ln(t-1)_{t=0} + \sum_{n=0}^{\infty} \frac{1}{n+1} t^{n+1}, \quad (|t| < 1)$$

± 1 为 $\ln \frac{t+1}{t-1}$ 的两个枝点，以连接两点的线段为割线，规定割线上岸

$$\arg(t+1) - \arg(t-1) = (2k+1)\pi, \quad (k=0, \pm 1, \pm 2, \dots), \text{ 则}$$

$$f\left(\frac{1}{t}\right) = \ln \frac{t+1}{t-1} \bigg|_{t=0} + \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{n+1} t^{n+1} = (2k+1)\pi i + \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n+1},$$

$$f(z) = (2k+1)\pi i + \sum_{n=0}^{\infty} \frac{2}{2n+1} z^{-(2n+1)}.$$

29. 验证等式 $\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\cdots=\int_0^1\frac{t^{a-1}}{1+t^b}dt$, ($a>0, b>0$)。因此, 此

类无穷级数求和就化为求定积分。利用这个办法求下列级数之和:

(1) $1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots$; (2) $\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\cdots$; (3) $1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\cdots$ 。

证: $\frac{t^{a-1}}{1+t^b}=\sum_{n=0}^{\infty}(-1)^nt^{a+bn-1}$, ($|t|<1$)

$$\int_0^xt^{\frac{a-1}{1+t^b}}dt=\sum_{n=0}^{\infty}(-1)^n\int_0^xt^{a+bn-1}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{a+bn}x^{a+bn},\quad(|x|<1)$$

由于级数 $\sum_{n=0}^{\infty}(-1)^nt^{a+bn-1}$ 在区间 $|t|\leq|x|$ ($|x|<1$) 上一致收敛, 所以上式中求和与积分可交

换顺序。上式右边的幂级数在 $x=1$ 点是收敛的, 所以它在 $x=1$ 点左连续 (Abel 第二定理),

即 $\lim_{x\rightarrow 1-0}\sum_{n=0}^{\infty}\frac{(-1)^n}{a+bn}x^{a+bn}=\sum_{n=0}^{\infty}\frac{(-1)^n}{a+bn}$ 。左边的积分是关于 x 的连续函数, 所以

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{a+bn}=\lim_{x\rightarrow 1-0}\sum_{n=0}^{\infty}\frac{(-1)^n}{a+bn}x^{a+bn}=\lim_{x\rightarrow 1-0}\int_0^x\frac{t^{a-1}}{1+t^b}dt=\int_0^1\frac{t^{a-1}}{1+t^b}dt。$$

(1) 这里 $a=1, b=3$, 所以和为

$$\int_0^1\frac{1}{1+t^3}dt=\frac{1}{3}\int_0^1\left(\frac{1}{t+1}-\frac{t-2}{t^2-t+1}\right)dt=\frac{1}{3}\int_0^1\frac{dt}{t+1}-\frac{1}{6}\int_0^1\frac{2t-1}{t^2-t+1}dt+\frac{1}{2}\int_0^1\frac{1}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}dt$$

$$=\frac{1}{3}\ln(t+1)\Big|_{t=0}^{t=1}-\frac{1}{6}\int_0^1\frac{d(t^2-t+1)}{t^2-t+1}+\frac{1}{\sqrt{3}}\int_0^1\frac{d\left[\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right]}{1+\left[\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right]^2}$$

$$=\frac{1}{3}\ln 2-\frac{1}{6}\ln(t^2-t+1)\Big|_{t=0}^{t=1}+\frac{1}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right]\Big|_{t=0}^{t=1}$$

$$=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\ln 2\right)$$

(2) $a=2, b=3$, 和为 $\int_0^1\frac{t}{1+t^3}dt$ 。由于

$$\int_0^1\frac{t}{1+t^3}dt+\int_0^1\frac{1}{1+t^3}dt=\int_0^1\frac{1}{t^2-t+1}dt=\frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right]\Big|_{t=0}^{t=1}=\frac{2\pi}{3\sqrt{3}}$$

$$\text{所以}\int_0^1\frac{t}{1+t^3}dt=\frac{2\pi}{3\sqrt{3}}-\int_0^1\frac{1}{1+t^3}dt=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}-\ln 2\right)。$$

(3) $a=1, b=4$, 和为 $\int_0^1\frac{1}{1+t^4}dt=\frac{1}{2\sqrt{2}}\int_0^1\left(\frac{t+\sqrt{2}}{t^2+\sqrt{2}t+1}-\frac{t-\sqrt{2}}{t^2-\sqrt{2}t+1}\right)dt$

$$=\frac{1}{4\sqrt{2}}\int_0^1\frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}dt+\frac{1}{2\sqrt{2}}\int_0^1\frac{d\left[\sqrt{2}\left(t+\frac{1}{\sqrt{2}}\right)\right]}{1+\left[\sqrt{2}\left(t+\frac{1}{\sqrt{2}}\right)\right]^2}-\frac{1}{4\sqrt{2}}\int_0^1\frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}dt+\frac{1}{2\sqrt{2}}\int_0^1\frac{d\left[\sqrt{2}\left(t-\frac{1}{\sqrt{2}}\right)\right]}{1+\left[\sqrt{2}\left(t-\frac{1}{\sqrt{2}}\right)\right]^2}$$

$$=\frac{1}{4\sqrt{2}}\ln(2+\sqrt{2})+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}+1)-\frac{\pi}{8\sqrt{2}}-\frac{1}{4\sqrt{2}}\ln(2-\sqrt{2})+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}-1)+\frac{\pi}{8\sqrt{2}}$$

$$=\frac{1}{4\sqrt{2}}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}+\frac{1}{2\sqrt{2}}\left[\arctan(\sqrt{2}+1)+\operatorname{arc}\cot(\sqrt{2}+1)\right]$$

$$=\frac{1}{4\sqrt{2}}\ln(3+2\sqrt{2})+\frac{1}{2\sqrt{2}}\cdot\frac{\pi}{2}=\frac{1}{4\sqrt{2}}\left[\ln(3+2\sqrt{2})+\pi\right]$$

30. 如果 k 和 n 是自然数, $a>0, b>0$, 证明:

15.4)

$$\frac{k!}{(a+nb)(a+nb+1)\cdots(a+nb+k)}=\int_0^1t^{a+nb-1}(1-t)^kdt, \text{ 并求下列级数之和:}$$

$$(1) \frac{1}{1\cdot 2\cdot 3}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}+\cdots; (2) \frac{1}{1\cdot 2\cdot 3}-\frac{1}{3\cdot 4\cdot 5}+\frac{1}{5\cdot 6\cdot 7}-+\cdots;$$

$$(3) \frac{1}{1\cdot 2\cdot 3}+\frac{1}{4\cdot 5\cdot 6}+\frac{1}{7\cdot 8\cdot 9}+\cdots; (4) \frac{1}{2\cdot 3\cdot 4}-\frac{1}{4\cdot 5\cdot 6}+\frac{1}{6\cdot 7\cdot 8}-+\cdots。$$

$$\text{证: } \int_0^xt^{p-1}(x-t)^kdt=\frac{1}{p}\int_0^x(t^p)'(x-t)^kdt=\frac{k}{p}\int_0^xt^p(x-t)^{k-1}dt$$

$$=\frac{k(k-1)}{p(p+1)}\int_0^xt^{p+1}(x-t)^{k-2}dt$$

$$=\cdots\cdots$$

$$=\frac{k!x^k}{p(p+1)\cdots(p+k)}x^p$$

$$\text{令 } p=a+nb, \quad x=1 \text{ 即可得 } \frac{k!}{(a+nb)(a+nb+1)\cdots(a+nb+k)}=\int_0^1t^{a+nb-1}(1-t)^kdt。$$

$$\text{级数}\sum_{n=0}^{\infty}\int_0^xt^{a+nb-1}(x-t)^kdt=x^{k+a}\sum_{n=0}^{\infty}\frac{k!}{(a+bn)(a+bn+1)\cdots(a+bn+k)}x^{bn} \text{ 是幂级数。}$$

$$\text{当 } k\geq 1 \text{ 时, } \sum_n\int_0^xt^{a+nb-1}(x-t)^kdt\Big|_{x=1}=\sum_n\frac{k!}{(a+nb)(a+nb+1)\cdots(a+nb+k)} \text{ 显然是收敛}$$

的, 由 Abel 第二定理, 幂级数 $\sum_n\int_0^xt^{a+nb-1}(x-t)^kdt$ 在 $x=1$ 处左连续, 即下式成立:

$$\lim_{x\rightarrow 1-0}\sum_n\int_0^xt^{a+nb-1}(x-t)^kdt=\sum_n\int_0^1t^{a+nb-1}(1-t)^kdt \quad (|x|<1)。$$

$$\text{级数}\sum_{n=0}^{\infty}\frac{1}{(a+nb)(a+nb+1)\cdots(a+nb+k)}=\frac{1}{k!}\sum_{n=0}^{\infty}\int_0^1t^{a+nb-1}(1-t)^kdt$$

$$=\frac{1}{k!}\lim_{x\rightarrow 1-0}\sum_{n=0}^{\infty}\int_0^xt^{a+nb-1}(x-t)^kdt$$

由于级数 $\sum_nt^{bn}$ 收敛半径为 1, 所以对于 $|x|<1$, 求和与积分可交换顺序, 即

$$\text{上式}=\frac{1}{k!}\lim_{x\rightarrow 1-0}\int_0^xt^{a-1}(x-t)^k\sum_{n=0}^{\infty}t^{bn}dt=\frac{1}{k!}\int_0^1\frac{t^{a-1}(1-t)^k}{1-t^b}dt$$

$$\text{同样可得}\sum_{n=0}^{\infty}\frac{(-1)^n}{(a+nb)(a+nb+1)\cdots(a+nb+k)}=\frac{1}{k!}\int_0^1\frac{t^{a-1}(1-t)^k}{1-t^b}dt。$$

$$(1) \text{ 这里 } a=1, b=2, k=2。 \text{ 和为 } \frac{1}{2}\int_0^1\frac{(1-t)^2}{1-t^2}dt。$$

$$\frac{1}{2}\int_0^1\frac{(1-t)^2}{1-t^2}dt=\frac{1}{2}\int_0^1\left(\frac{2}{1+t}-1\right)dt=\ln 2-\frac{1}{2};$$

$$(2) \text{ 和为 } \frac{1}{2}\int_0^1\frac{(1-t)^2}{1+t^2}dt=\frac{1}{2}\int_0^1\left(1-\frac{2t}{1+t^2}\right)dt=\frac{1}{2}(1-\ln 2)。$$

$$(3) a=1, b=3, k=2。 \text{ 和为 } \frac{1}{2}\int_0^1\frac{(1-t)^2}{1-t^3}dt$$

$$=\frac{\sqrt{3}}{2}\int_0^1\frac{d\left[\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right]}{1+\left[\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right]^2}-\frac{1}{4}\int_0^1\frac{2t+1}{t^2+t+1}dt=\frac{1}{4}\left(\frac{\pi}{\sqrt{3}}-\ln 3\right)$$

$$(4) a=2, b=2, k=2。 \text{ 和为 } \frac{1}{2}\int_0^1\frac{t(1-t)^2}{1+t^2}dt=\int_0^1\left(\frac{1}{2}t-1+\frac{1}{1+t^2}\right)dt=\frac{1}{4}(\pi-3)。$$

31. 求下列函数的 Laurent 展开: (1) $\frac{1}{z^2(z-1)}$, 在 $z=1$ 附近展开;

16.1)

长留数定理

(2) $\frac{1}{z^2-3z+2}$, 展开区域为: (i) $1<|z|<2$, (ii) $2<|z|<\infty$;

(3) $\frac{1}{z(z+1)}$, 展开区域为: (i) $1<|z-i|<\sqrt{2}$, (ii) $0<|z|<1$;

(4) $\frac{(z-1)(z-2)}{(z-3)(z-4)}$, 展开区域为: (i) $3<|z|<4$, (ii) $4<|z|<\infty$;

(5) $\frac{e^z}{z+2}$, 在 $|z|>2$ 处展开; (6) $\frac{1}{1-\cos z}$, 在 $z=2n\pi$ 附近展开 (可只求出不为 0 的

前四项系数)。

(1) $\frac{1}{z^2} = [1+(z-1)]^{-2} = \sum_{n=0}^{\infty} \frac{(-2)(-2-1)(-2-2)\cdots(-2-n+1)}{n!} (z-1)^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} (z-1)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) (z-1)^n \quad (|z-1|<1)$$

$\frac{1}{z^2(z-1)} = (z-1)^{-1} \sum_{n=0}^{\infty} (-1)^n (n+1) (z-1)^n = \sum_{k=1}^{\infty} (-1)^{k+1} (k+2) (z-1)^k, \quad (0<|z-1|<1)$

(2) (i) $\frac{1}{z^2-3z+2} = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2} \frac{1}{1-z/2} - z^{-1} \frac{1}{1-z^{-1}}$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} z^{-n} - z^{-1} \sum_{n=0}^{\infty} z^{-n} = -\sum_{n=1}^{\infty} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^{-n} \quad (1<|z|<2)$$

(ii) $\frac{1}{z^2-3z+2} = z^{-1} \frac{1}{1-2z^{-1}} - z^{-1} \frac{1}{1-z^{-1}} = \sum_{k=2}^{\infty} (2^{k-1}-1) z^k \quad (2<|z|<\infty)$

(3) (i) $\frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1} = (z-i)^{-1} \frac{1}{1+\frac{i}{z-i}} - \frac{1}{1+i} \frac{1}{1+\frac{z-i}{1+i}}$

$$= \sum_{n=1}^{\infty} i^{n+1} (z-i)^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} (z-i)^n \quad (1<|z-i|<\sqrt{2})$$

(ii) $\frac{1}{z(z+1)} = z^{-1} - \frac{1}{1+z} = z^{-1} + \sum_{n=1}^{\infty} (-1)^{n+1} z^n = \sum_{n=1}^{\infty} (-1)^{n+1} z^n \quad (0<|z|<1)$

(4) (i) $\frac{(z-1)(z-2)}{(z-3)(z-4)} = 1 + \frac{6}{z-4} - \frac{2}{z-3} = 1 - \frac{3}{2} \frac{1}{1-z/4} - 2z^{-1} \frac{1}{1-3z^{-1}}$

$$= 1 - \frac{3}{2} \sum_{n=0}^{\infty} 4^{-n} z^{-n} - 2 \sum_{n=1}^{\infty} 3^{-n-1} z^n \quad (3<|z|<4)$$

(ii) $\frac{(z-1)(z-2)}{(z-3)(z-4)} = 1 + 6z^{-1} \frac{1}{1-4z^{-1}} - 2z^{-1} \frac{1}{1-3z^{-1}}$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{3}{2} \cdot 4^{-n} - \frac{2}{3} \cdot 3^{-n} \right) z^n \quad (4<|z|<\infty)$$

(5) $\frac{1}{z+2} = z^{-1} \frac{1}{1+2/z} = \sum_{n=0}^{\infty} (-2)^n z^{-n-1} \quad (|z|>2), \quad e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad (|z|<\infty)$, 两级数都

是绝对收敛的, 所以可用 Cauchy 乘积计算 $\frac{e^z}{z+2}$ 。

$$\begin{aligned} \frac{e^z}{z+2} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{z^k}{k!} \cdot (-2)^{n-k} z^{k-n-1} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(-2)^{n-k}}{k!} z^{2k-n-1} \\ &= \sum_{k=0}^{\infty} \sum_{n=2k}^{\infty} \frac{(-2)^{n-k}}{k!} z^{2k-n-1} + \sum_{k=0}^{\infty} \sum_{n=k}^{2k-1} \frac{(-2)^{n-k}}{k!} z^{2k-n-1} \\ &= \sum_{k=0}^{\infty} \sum_{m=n-1}^{-\infty} \frac{(-2)^{k-m-1}}{k!} z^m + \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(-2)^{k-m-1}}{k!} z^m \\ &= \sum_{m=-1}^{\infty} \sum_{k=0}^{\infty} \frac{(-2)^{k-m-1}}{k!} z^m + \sum_{m=0}^{\infty} \sum_{k=m+1}^{\infty} \frac{(-2)^{k-m-1}}{k!} z^m \quad (2<|z|<\infty) \end{aligned}$$

(6) $\frac{1}{1-\cos z} = \frac{1}{1-\cos(z-2n\pi)} = \frac{1}{2\sin^2\left(\frac{z-2n\pi}{2}\right)}$, 可看出 $z=2n\pi$ 是二阶极点, 且它

是关于 $z-2n\pi$ 的偶函数, 故可设 $\frac{1}{1-\cos z} = \sum_{n=-1}^{\infty} a_n (z-2n\pi)^{2n}$ 。

$1 = (1-\cos z) \sum_{n=-1}^{\infty} a_n (z-2n\pi)^{2n} = \sum_{n=1}^{\infty} b_n (z-2n\pi)^{2n} \cdot \sum_{n=-1}^{\infty} a_n (z-2n\pi)^{2n}$, 其中 $b_n = \frac{(-1)^{n+1}}{(2n)!}$ 。

假设上式右边的 Cauchy 乘积收敛, 则 $1 = \sum_{n=0}^{\infty} \sum_{k=-1}^{n+1} a_k b_{n-k} (z-2n\pi)^{2n}$ 。比较系数可得

$1 = a_{-1} b_1, \quad 0 = a_{-1} b_2 + a_0 b_1, \quad 0 = a_{-1} b_3 + a_0 b_2 + a_1 b_1, \quad 0 = a_{-1} b_4 + a_0 b_3 + a_1 b_2 + a_2 b_1, \quad \cdots$

解得 $a_{-1} = 2, \quad a_0 = \frac{1}{6}, \quad a_1 = \frac{1}{120}, \quad a_2 = \frac{1}{3024}, \quad \cdots$

即 $\frac{1}{1-\cos z} = 2(z-2n\pi)^{-2} + \frac{1}{6} + \frac{1}{120}(z-2n\pi)^2 + \frac{1}{3024}(z-2n\pi)^4 + \cdots \quad (0<|z-2n\pi|<2\pi)$

5. 判断下列函数奇点的性质, 如果是极点, 确定其阶数: (1) $\frac{1}{z^2+a^2}$; (2) $\frac{\cos az}{z^2}$;

(3) $\frac{\cos az - \cos bz}{z^2}$; (4) $\frac{\sin z}{z^2} - \frac{1}{z}$; (5) $\frac{1}{e^z-1} - \frac{1}{z}$; (6) $\sin \frac{1}{z}$; (7) $\frac{\sqrt{z}}{\sin \sqrt{z}}$;

(8) $\int_0^z \frac{e^{\sqrt{\zeta}} - e^{-\sqrt{\zeta}}}{\sqrt{\zeta}} d\zeta$ 。

17.1)

(1) $\pm ai$ 是一阶极点。 $f\left(\frac{1}{t}\right) = \frac{t^2}{1+a^2 t^2}$, 0 不是 $f(1/t)$ 的奇点, 故 ∞ 不是 $f(z)$ 的奇点。

(2) 0 是二阶极点: $f\left(\frac{1}{t}\right) = t^2 \left[1 - \frac{1}{2!} \left(\frac{a}{t}\right)^2 + \frac{1}{4!} \left(\frac{a}{t}\right)^4 - \cdots \right] = t^2 - \frac{a^2}{2!} + \frac{a^4}{4!} t^{-2} - \cdots$

0 是 $f(1/t)$ 的本性奇点, 所以 ∞ 是 $f(z)$ 的本性奇点。

(3) $f(z) = z^{-2} \left[\left(1 - \frac{1}{2!} a^2 z^2 + \frac{1}{4!} a^4 z^4 - \cdots \right) - \left(1 - \frac{1}{2!} b^2 z^2 + \frac{1}{4!} b^4 z^4 - \cdots \right) \right]$

$$= -\frac{1}{2!} (a^2 - b^2) + \frac{1}{4!} (a^4 - b^4) z^2 - \cdots$$

所以 0 是 $f(z)$ 的可去奇点。 $f(1/t) = -\frac{1}{2!} (a^2 - b^2) + \frac{1}{4!} (a^4 - b^4) t^{-2} - \cdots$, ∞ 是 $f(z)$ 的本性奇点。

(4) $f(z) = z^{-2} \left(z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \cdots \right) - z^{-1} = -\frac{1}{3!} z + \frac{1}{5!} z^3 - \cdots$, 0 是可去奇点。

$f\left(\frac{1}{t}\right) = -\frac{1}{3!} t^{-1} + \frac{1}{5!} t^{-3} - \cdots$, ∞ 是 $f(z)$ 的本性奇点。

(5) $\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \frac{z - e^z + 1}{z(e^z - 1)} = \lim_{z \rightarrow 0} \frac{z + 1 - (1 + z + z^2/2 + \cdots)}{z^2} = -\frac{1}{2}$, 所以 0 是可去奇点。

$\lim_{z \rightarrow 2n\pi i} (z - 2n\pi i) f(z) = \lim_{z \rightarrow 2n\pi i} \frac{(z - 2n\pi i)(z - e^z + 1)}{z(e^z - 1)} = \lim_{z \rightarrow 2n\pi i} \frac{z - e^z + 1 + (z - 2n\pi i)(1 - e^z)}{e^z - 1 + ze^z} = 1$

所以 $2n\pi i$ ($n = \pm 1, \pm 2, \cdots$) 是一阶奇点。由于 $2n\pi i \rightarrow \infty$, 所以在 ∞ 点的任一邻域内有无穷个奇点, 即 ∞ 是非孤立奇点。

(6) $f(z) = z^{-1} - \frac{1}{3!} z^{-3} + \frac{1}{5!} z^{-5} - \cdots$, 所以 0 是本性奇点。

$f(1/t) = t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \cdots$, 所以 ∞ 点是可去奇点。

(7) $\lim_{z \rightarrow 0} f(z) = 1$, 所以 0 是可去奇点。

$$\lim_{z \rightarrow n^2 \pi^2} (z - n^2 \pi^2) f(z) = \lim_{z \rightarrow n^2 \pi^2} \frac{(z - n^2 \pi^2) \sqrt{z}}{\sin \sqrt{z}} = \lim_{z \rightarrow n^2 \pi^2} \frac{\sqrt{z} + \frac{(z - n^2 \pi^2)}{2\sqrt{z}}}{\frac{\cos \sqrt{z}}{2\sqrt{z}}} = (-1)^n 2n^2 \pi^2$$

所以 $n^2 \pi^2$ ($n = 1, 2, 3, \cdots$) 是一阶极点。由于 $n^2 \pi^2 \rightarrow \infty$, 所以 ∞ 点的任意邻域包含无穷个奇点, 即 ∞ 点为非孤立奇点。

(8) $f(z) = \int_0^z \frac{e^{\sqrt{\zeta}} - e^{-\sqrt{\zeta}}}{\sqrt{\zeta}} d\zeta = \int_0^z \zeta^{-\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{\zeta^{n/2}}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{n/2}}{n!} \right) d\zeta$

$$= 2 \int_0^z \sum_{k=0}^{\infty} \frac{\zeta^{k+1/2}}{(2k+1)!} d\zeta = 2 \sum_{k=0}^{\infty} \frac{\zeta^{k+1}}{(k+1)(2k+1)!}$$

$f(1/t) = 2 \sum_{k=0}^{\infty} \frac{t^{-(k+1)}}{(k+1)(2k+1)!}$, 即 ∞ 点为本性奇点。

$f(z) = 2 \int_0^z (e^{\sqrt{\zeta}} - e^{-\sqrt{\zeta}}) d\sqrt{\zeta} = 2(e^{\sqrt{z}} + e^{-\sqrt{z}} - 2)$, 即除 ∞ 点外无其他奇点。

33. 求下列函数在指定点 z_0 的留数: (1) $\frac{e^{z^2}}{z-1}$, $z_0=1$; (2) $\frac{e^{z^2}}{(z-1)^2}$, $z_0=1$;

(3) $\left(\frac{z}{1-\cos z}\right)^2$, $z_0=0$; (4) $\frac{z^2}{z^4-1}$, $z_0=i$; (5) $\frac{1}{z^2 \sin z}$, $z_0=0$; (6) $\frac{1+e^z}{z^4}$, $z_0=0$;

(7) $\frac{e^z}{(z^2-1)^2}$, $z_0=1$; (8) $\frac{1}{\operatorname{ch} \sqrt{z}}$, $z_0=-\left(\frac{2n+1}{2}\pi\right)^2$, $n=0,1,2,\dots$ 。

(1) $\operatorname{res} f(1)=\lim_{z \rightarrow 1}(z-1)f(z)=e$ 。

(2) $\operatorname{res} f(1)=\lim_{z \rightarrow 1} \frac{d}{dz}[(z-1)^2 f(z)]=2e$ 。

(3) $1-\cos z=1-\left(1-\frac{1}{2!}z^2+\frac{1}{4!}z^4-\frac{1}{6!}z^6+\dots\right)=\frac{1}{2}z^2\left(1-\frac{1}{12}z^2+\frac{1}{360}z^4-\dots\right)$,

$$\begin{aligned}\left(\frac{z}{1-\cos z}\right)^2 &= 4z^{-2}\left(1-\frac{1}{12}z^2+\frac{1}{360}z^4-\dots\right)^{-2}=4z^{-2}\left[1+2\left(\frac{1}{12}z^2-\frac{1}{360}z^4+\dots\right)+O(z^4)\right] \\ &= 4z^{-2}+\frac{2}{3}-\frac{1}{45}z^2+\dots\end{aligned}$$

所以 $\operatorname{res} f(0)=0$ 。

(4) $f(z)=\frac{z^2}{(z+i)(z-i)(z+1)(z-1)}$, i 是一阶极点, $\operatorname{res} f(i)=\lim_{z \rightarrow i}(z-i)f(z)=-\frac{1}{4}i$ 。

(5) 显然 0 是三阶极点。 $\operatorname{res} f(0)=\frac{1}{2}\lim_{z \rightarrow 0} \frac{d^2}{dz^2} z^3 f(z)=\frac{1}{2}\lim_{z \rightarrow 0} \frac{z(1+\cos^2 z)-\sin 2z}{\sin^3 z}$

$$=\frac{1}{2}\lim_{z \rightarrow 0} \frac{3\sin^2 z-2z \sin z \cos z}{3\sin^2 z \cos z}=\frac{1}{2}\lim_{z \rightarrow 0} \left(\frac{1}{\cos z}-\frac{2z}{3\sin z}\right)=\frac{1}{6}$$

(6) $\operatorname{res} f(0)=\frac{1}{3!}\lim_{z \rightarrow 0} \frac{d^3}{dz^3} z^4 f(z)=\frac{1}{6}\lim_{z \rightarrow 0} e^z=\frac{1}{6}$ 。

(7) $\operatorname{res} f(1)=\lim_{z \rightarrow 1} \frac{d}{dz}(z-1)^2 f(z)=0$ 。

(8) $\operatorname{res} f\left[-\left(\frac{2n+1}{2}\pi\right)^2\right]=\lim_{z \rightarrow -\left(\frac{2n+1}{2}\pi\right)^2}\left[z+\left(\frac{2n+1}{2}\pi\right)^2\right]f(z)$

$$=\lim_{z \rightarrow -\left(\frac{2n+1}{2}\pi\right)^2} \frac{2\left[z+\left(\frac{2n+1}{2}\pi\right)^2\right]}{e^{i\sqrt{z}}+e^{-i\sqrt{z}}}=\lim_{z \rightarrow -\left(\frac{2n+1}{2}\pi\right)^2} \frac{4\sqrt{z}}{e^{i\sqrt{z}}-e^{-i\sqrt{z}}}=(-1)^n(2n+1)\pi$$

34 求下列函数在奇点处的留数: (1) $\frac{1}{z^3-z^5}$; (2) $\frac{1}{(1+z^2)^{m+1}}$; (3) $\frac{z}{1-\cos z}$;

(4) $\frac{\sqrt{z}}{\operatorname{sh} \sqrt{z}}$; (5) $e^{\frac{1}{1-z}}$; (6) $\cos \sqrt{\frac{1}{z}}$; (7) $\frac{1}{(z-1)\ln z}$;

(8) $\frac{1}{z}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\dots+\frac{1}{(z+1)^n}\right]$ 。

(1) $f(z)=\frac{1}{z^3(z+1)(z-1)}$, 0 是三阶极点, ± 1 是一阶极点。 $f(1/t)=\frac{t^5}{t^2-1}$, ∞ 点

是可去奇点。 $\operatorname{res} f(0)=\frac{1}{2}\lim_{z \rightarrow 0} \frac{d^2}{dz^2} z^3 f(z)=1$, $\operatorname{res} f(1)=\lim_{z \rightarrow 1}(z-1)f(z)=\frac{1}{2}$,

$\operatorname{res} f(-1)=\lim_{z \rightarrow -1}(z+1)f(z)=\frac{1}{2}$ 。

(2) $\pm i$ 是 $m+1$ 阶极点, ∞ 点是可去奇点。

$$\begin{aligned}\operatorname{res} f(i) &= \frac{1}{m!}\lim_{z \rightarrow i} \frac{d^m}{dz^m} \frac{1}{(z+i)^{m+1}}=\frac{1}{m!}\lim_{z \rightarrow i} \frac{(-1)^m(m+1)(m+2)\cdots(2m)}{(z+i)^{2m+1}} \\ &= \frac{1}{(m!)^2} \frac{(-1)^m(2m)!}{(2i)^{2m+1}}=-i \frac{(2m)!}{(m!)^2 2^{2m+1}}\end{aligned}$$

$$\operatorname{res} f(-i)=i \frac{(2m)!}{(m!)^2 2^{2m+1}}$$

(3) 可看出 0 是一阶极点, $2n\pi$ ($n=\pm 1, \pm 2, \dots$) 是二阶极点, ∞ 点是非孤立奇点。

$$\operatorname{res} f(0)=\lim_{z \rightarrow 0} \frac{z^2}{1-\cos z}=\lim_{z \rightarrow 0} \frac{z^2}{2\sin^2 \frac{z}{2}}=2。$$

$$1-\cos z=1-\cos(z-2n\pi)=\frac{1}{2}(z-2n\pi)^2\left[1-\frac{1}{12}(z-2n\pi)^2+\frac{1}{360}(z-2n\pi)^4-\dots\right]$$

$$\begin{aligned}\frac{z}{1-\cos z} &= (z-2n\pi)\left[1-\cos(z-2n\pi)\right]^{-1}+2n\pi\left[1-\cos(z-2n\pi)\right]^{-1} \\ &= 2(z-2n\pi)^{-1}\left[1+\frac{1}{12}(z-2n\pi)^2+O(z-2n\pi)^4\right] \\ &\quad +4n\pi(z-2n\pi)^{-2}\left[1+\frac{1}{12}(z-2n\pi)^2+O(z-2n\pi)^4\right] \\ &= 4n\pi(z-2n\pi)^{-2}+2(z-2n\pi)^{-1}+\frac{n\pi}{3}+\frac{1}{6}(z-2n\pi)+\dots\end{aligned}$$

即 $\operatorname{res} f(2n\pi)=2$ 。

(4) $f(z)=\frac{2\sqrt{z}}{e^{\sqrt{z}}-e^{-\sqrt{z}}}$, 奇点是 $-k^2\pi^2$ ($k=0, \pm 1, \pm 2, \dots$), ∞ 点是非孤立奇点。

$\lim_{z \rightarrow 0} f(z)=\lim_{z \rightarrow 0} \frac{2}{e^{\sqrt{z}}+e^{-\sqrt{z}}}=1$, 所以 0 是可去奇点, $\operatorname{res} f(0)=0$ 。 $k \neq 0$ 时,

$$\operatorname{res} f(-k^2\pi^2)=\lim_{z \rightarrow -k^2\pi^2} \frac{2(z+k^2\pi^2)\sqrt{z}}{e^{\sqrt{z}}-e^{-\sqrt{z}}}=\lim_{z \rightarrow -k^2\pi^2} \frac{4z+2(z+k^2\pi^2)}{e^{\sqrt{z}}+e^{-\sqrt{z}}}=(-1)^{k+1}2k^2\pi^2。$$

(5) $f(z)=1+\frac{1}{1-z}+\frac{1}{2!}\left(\frac{1}{1-z}\right)^2+\frac{1}{3!}\left(\frac{1}{1-z}\right)^3+\dots=1-(z-1)^{-1}+\frac{1}{2}(z-1)^{-2}-\frac{1}{3!}(z-1)^{-3}+\dots$

即 $\operatorname{res} f(1)=-1$ 。由于 $f(z)$ 只有两个孤立奇点 $1, \infty$, 所以 $\operatorname{res} f(\infty)=-\operatorname{res} f(1)=1$ 。

(6) $f(z)=1-\frac{1}{2!}\left(\sqrt{\frac{1}{z}}\right)^2+\frac{1}{4!}\left(\sqrt{\frac{1}{z}}\right)^4-\dots=1-\frac{1}{2}z^{-1}+\frac{1}{24}z^{-2}-\dots$

即 $\operatorname{res} f(0)=-\frac{1}{2}$ 。 $\operatorname{res} f(\infty)=-\operatorname{res} f(0)=\frac{1}{2}$ 。

(7) 若规定 $\ln z|_{z=1}=0$, 则 1 是二阶极点。此时

$$\operatorname{res} f(1)=\lim_{z \rightarrow 1} \frac{d}{dz}(z-1)^2 f(z)=\lim_{z \rightarrow 1} \frac{z \ln z - z + 1}{z(\ln z)^2}=\lim_{z \rightarrow 1} \frac{1}{\ln z + 2}=\frac{1}{2}。$$

若规定 $\ln z|_{z=1}=2k\pi i$ ($k=\pm 1, \pm 2, \dots$), 则 1 是一阶极点。此时

$$\operatorname{res} f(1)=\lim_{z \rightarrow 1}(z-1)f(z)=\lim_{z \rightarrow 1} \frac{1}{\ln z}=\frac{1}{2k\pi i}。$$

(8) $\operatorname{res} f(0)=\lim_{z \rightarrow 0} z f(z)=n+1$ 。

$$\begin{aligned}\operatorname{res} f(-1) &= \frac{1}{(n-1)!}\lim_{z \rightarrow -1} \frac{d^{n-1}}{dz^{n-1}}[(z+1)^n f(z)] \\ &= \frac{1}{(n-1)!}\lim_{z \rightarrow -1} \frac{d^{n-1}}{dz^{n-1}}\left\{\frac{1}{z}\left[(z+1)^n+(z+1)^{n-1}+\dots+(z+1)+1\right]\right\} \\ &= \frac{1}{(n-1)!}\lim_{z \rightarrow -1} \frac{d^{n-1}}{dz^{n-1}}\left[\frac{1}{z}\left(z^n+a_{n-1}z^{n-1}+\dots+a_1z+n+1\right)\right] \\ &= \frac{1}{(n-1)!}\lim_{z \rightarrow -1} \frac{d^{n-1}}{dz^{n-1}}\left(z^{n-1}+a_{n-1}z^{n-2}+\dots+a_1+\frac{n+1}{z}\right) \\ &= \frac{1}{(n-1)!}\lim_{z \rightarrow -1}\left[(n-1)!+0+\dots+0+(n+1)\frac{(-1)^{n-1}(n-1)!}{z^n}\right] \\ &=-n\end{aligned}$$

$\operatorname{res} f(\infty)=-\operatorname{res} f(0)-\operatorname{res} f(-1)=-1$ 。

35 指出下列函数在 ∞ 点的性质, 并求其留数: (1) $\frac{1}{z}$; (2) $\frac{\cos z}{z}$; (3) $\frac{z}{\cos z}$;

(4) $\frac{z^2+1}{e^z}$; (5) $e^{-\frac{1}{z^2}}$; (6) $\sqrt{(z-1)(z-2)}$ 。 **(7.2)**

(1) $f(1/t)=t$, 所以 ∞ 是可去奇点。 ∞ 点的留数等于 $-f(1/t)$ 在 $t=0$ 点邻域内幂级数

展开中 t 的系数, 即 $\operatorname{res} f(\infty)=-1$ 。

$$(2) f(1/t) = t \cos \frac{1}{t} = t \left(1 - \frac{1}{2!t^2} + \frac{1}{4!t^4} - + \cdots \right) = t - \frac{1}{2}t^{-1} + \frac{1}{24}t^{-3} - + \cdots,$$

即 ∞ 是本性奇点, $\operatorname{res} f(\infty) = -1$ 。

$$(3) \text{ 易知 } z = \left(k + \frac{1}{2}\right)\pi \text{ 是奇点, 由于 } \left(k + \frac{1}{2}\right)\pi \rightarrow \infty, \text{ 所以 } \infty \text{ 是非孤立奇点。}$$

$$(4) f(1/t) = \left(\frac{1}{t^2} + 1\right)e^{-t^{-1}} = t^{-2} \left(1 - t^{-1} + \frac{1}{2}t^{-2} - + \cdots\right) + \left(1 - t^{-1} + \frac{1}{2}t^{-2} - + \cdots\right) \\ = 1 - t^{-1} + \frac{3}{2}t^{-2} + \cdots$$

即 ∞ 是本性奇点, $\operatorname{res} f(\infty) = 0$ 。

$$(5) f(1/t) = e^{-t^2}, \quad \infty \text{ 点为可去奇点, } \operatorname{res} f(\infty) = 0。$$

$$(6) f(1/t) = \frac{\sqrt{(1-t)(1-2t)}}{t}, \quad 0 \text{ 为 } f(1/t) \text{ 的一阶极点, 故 } \infty \text{ 点为 } f(z) \text{ 的一阶极点。}$$

1 和 1/2 是 $f(1/t)$ 的分枝点。规定 $\arg(1-t)_{t=0} = 2n\pi, \arg(1/2-t)_{t=0} = 2m\pi$, 则

$$\left(\sqrt{1-t}\right)_{t=0}^{(k)} = (-1)^k \frac{1}{2} \left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-k+1\right) \frac{\sqrt{1-t}}{(1-t)^k} \Big|_{t=0} = (-1)^n (-1)^k \frac{1}{2} \left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-k+1\right)$$

$$\left(\sqrt{1-2t}\right)_{t=0}^{(k)} = (-2)^k \frac{1}{2} \left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-k+1\right) \frac{\sqrt{1-2t}}{(1-2t)^k} \Big|_{t=0} = (-1)^m (-2)^k \frac{1}{2} \left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-k+1\right)$$

$$f(1/t) = (-1)^{n+m} t^{-1} \left(1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 + \cdots\right) \left(1 - t - \frac{1}{2}t^2 - \frac{1}{2}t^3 + \cdots\right) \\ = (-1)^{n+m} \left(t^{-1} - \frac{3}{2} - \frac{1}{8}t - \frac{7}{16}t^2 + \cdots\right)$$

即 $\operatorname{res} f(\infty) = (-1)^{n+m} \frac{1}{8}$ 。

36 计算下列积分: (1) $\int_0^{2\pi} \frac{dx}{1-2p \cos x + p^2}, \quad 0 < p < 1;$ (2) $\int_0^{2\pi} \frac{dx}{2 + \cos x};$

$$(3) \int_0^{2\pi} \frac{dx}{(a+b \cos x)^2}, \quad a > b > 0; (4) \int_0^{2\pi} \cos^{2n} x dx; (5) \int_0^{2\pi} \exp(e^{i\theta}) d\theta;$$

$$(6) \int_0^{\pi} \frac{d\theta}{1 + \sin^2 \theta}; (7) \int_0^{\pi} \frac{d\theta}{(1 + \sin^2 \theta)^2}; (8) \int_0^{\pi} \cot(x-\alpha) dx, \quad \operatorname{Im} \alpha \neq 0。$$

$$(1) \text{ 令 } f(z) = \frac{1}{z} \frac{1}{1-2p \frac{z^2+1}{2z} + p^2} = -\frac{1}{p} \frac{1}{(z-p) \left(z - \frac{1}{p}\right)}, \text{ 则 } \operatorname{res} f(p) = \frac{1}{1-p^2}。$$

$$\text{原积分} = 2\pi \operatorname{res} f(p) = \frac{2\pi}{1-p^2}。$$

$$(2) \text{ 令 } f(z) = \frac{1}{z} \frac{1}{2 + \frac{z^2+1}{2z}} = \frac{2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}, \text{ 则 } \operatorname{res} f(-2+\sqrt{3}) = \frac{1}{\sqrt{3}}。$$

$$\text{原积分} = \frac{2\pi}{\sqrt{3}}。$$

$$(3) \text{ 令 } f(z) = \frac{1}{z \left(a+b \frac{z^2+1}{2z}\right)^2} = \frac{4}{b^2} \frac{z}{(z-z_1)^2 (z-z_2)^2}, \text{ 其中 } z_1 = -\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1},$$

$$z_2 = -\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}, \quad -\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1} < -\frac{a}{b} < -1, \text{ 所以 } |z_1| > 1, \text{ 在单位圆外,}$$

$$|z_2| = \left| -\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1} \right| = 1 / \left| -\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1} \right| < 1, \text{ 在单位圆内。}$$

$$\operatorname{res} f(z_2) = \lim_{z \rightarrow z_2} \frac{d}{dz} \left[(z-z_2)^2 f(z) \right] = \frac{a}{(a^2-b^2)^{3/2}},$$

$$\text{所以原积分} = \frac{2\pi a}{(a^2-b^2)^{3/2}}。$$

$$(4) \text{ 令 } f(z) = \frac{1}{z} \left(\frac{z^2+1}{2z} \right)^{2n} = \frac{1}{2^{2n}} \frac{(1+z^2)^{2n}}{z^{2n+1}} = \frac{1}{2^{2n}} \frac{1}{z^{2n+1}} \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} z^{2k},$$

$$\operatorname{res} f(0) = \lim_{z \rightarrow 0} \left[\frac{1}{2^{2n}} \frac{d^{2n}}{dz^{2n}} \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} z^{2k} \right]$$

$$= \frac{1}{2^{2n}} \frac{d^{2n}}{(2n)! dz^{2n}} \left[1 + a_1 z + \cdots + \frac{(2n)!}{(n!)^2} z^{2n} + \cdots + z^{4n} \right]_{z=0}$$

$$= \frac{1}{2^{2n}} \frac{1}{(2n)!} \left[0 + 0 + \cdots + \frac{(2n)!}{(n!)^2} (2n)! + b_{2n+1} z + \cdots + b_{4n} z^{2n} \right]_{z=0} = \frac{(2n)!}{2^{2n} (n!)^2}$$

$$\text{所以原积分} = 2\pi \frac{(2n)!}{2^{2n} (n!)^2} = 2\pi \frac{2n(2n-1)(2n-2)(2n-3) \cdots 2 \cdot 1}{2^{2n} n^2 (n-1)^2 (n-2)^2 \cdots 2^2 \cdot 1^2}$$

$$= 2\pi \frac{2n(2n-1)(2n-2)(2n-3) \cdots 2 \cdot 1}{(2n)^2 (2n-2)^2 (2n-4)^2 \cdots 2^2} = 2\pi \frac{(2n-1)(2n-3) \cdots 3 \cdot 1}{(2n)(2n-2)(2n-4) \cdots 2}$$

$$= 2\pi \frac{(2n-1)!!}{(2n)!!}。$$

$$(5) \text{ 令 } f(z) = \frac{1}{z} e^z, \text{ 则 } \operatorname{res} f(0) = 1, \text{ 原积分} = 2\pi。$$

$$(6) \text{ 原积分} = \int_0^{\pi} \frac{2d\theta}{3 - \cos 2\theta} = \int_0^{2\pi} \frac{d\varphi}{3 - \cos \varphi}。$$

$$\text{令 } f(z) = \frac{1}{z} \frac{1}{3 - \frac{z^2+1}{2z}} = -\frac{2}{(z-3-2\sqrt{2})(z-3+2\sqrt{2})}, \text{ 则 } \operatorname{res} f(3-2\sqrt{2}) = \frac{1}{2\sqrt{2}}。$$

$$\text{原积分} = \frac{\pi}{\sqrt{2}}。$$

$$(7) \text{ 原积分} = \int_0^{\pi} \frac{4d\theta}{(3 - \cos 2\theta)^2} = \int_0^{2\pi} \frac{2d\varphi}{(3 - \cos \varphi)^2}。$$

$$\text{令 } f(z) = \frac{1}{z} \frac{2}{\left(3 - \frac{z^2+1}{2z}\right)^2} = \frac{8z}{(z-3-2\sqrt{2})^2 (z-3+2\sqrt{2})}, \text{ 则}$$

$$\operatorname{res} f(3-2\sqrt{2}) = \lim_{z \rightarrow 3-2\sqrt{2}} \frac{d}{dz} \frac{8z}{(z-3-2\sqrt{2})^2} = \frac{3}{8\sqrt{2}}, \text{ 原积分} = \frac{3}{4\sqrt{2}} \pi。$$

$$(8) \text{ 令 } \alpha = a + ib, \text{ 原积分} = i \int_0^{\pi} \frac{e^{i(x-a-ib)}}{e^{i(x-a-ib)} - e^{-i(x-a-ib)}} dx = i \int_0^{\pi} \frac{e^{i2(x-a-ib)} + 1}{e^{i2(x-a-ib)} - 1} dx$$

$$= \frac{i}{2} \int_{-2a}^{2\pi-2a} \frac{e^{2b} e^{i\theta} + 1}{e^{2b} e^{i\theta} - 1} d\theta \quad (\text{作代换 } 2(x-a) = \theta), \text{ 因为被积函数以 } 2\pi \text{ 为周期, 所以}$$

$$\text{原积分} = \frac{i}{2} \int_0^{2\pi} \frac{e^{2b} e^{i\theta} + 1}{e^{2b} e^{i\theta} - 1} d\theta = \frac{1}{2} \oint_{|z|=1} \frac{e^{2b} z + 1}{z(e^{2b} z - 1)} dz。$$

$$\text{记 } f(z) = \frac{z + e^{-2b}}{2z(z - e^{-2b})}, \text{ 则 } \operatorname{res} f(0) = -\frac{1}{2}, \quad \operatorname{res} f(e^{-2b}) = 1。$$

$$\text{若 } b > 0, \text{ 原积分} = 2\pi i \cdot [\operatorname{res} f(0) + \operatorname{res} f(e^{-2b})] = \pi i,$$

$$\text{若 } b < 0, \text{ 原积分} = 2\pi i \cdot \operatorname{res} f(0) = -\pi i。$$

37 计算下列积分: (1) $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx;$ (2) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a > 0, b > 0;$

$$(3) \int_{-\infty}^{\infty} \frac{x^{2m}}{x^{2n}+1} dx, \quad m, n \text{ 均为正整数, 且 } n > m; (4) \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}, \quad n \text{ 为正整数;}$$

$$(5) \int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)^2}, \quad a > 0; \quad (6) \int_{-\infty}^{\infty} \frac{dx}{x^2-2x+4};$$

$$(7) \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2-2x \cos \theta + 1)}, \quad \theta \text{ 为实数, 且 } \sin \theta \neq 0;$$

$$(8) \int_{-\infty}^{\infty} \frac{dx}{(1+x^2) \operatorname{ch} \frac{\pi x}{2}}。$$

$$(1) \text{ 令 } f(z) = \frac{z^2}{1+z^4} = \frac{z^2}{(z-e^{i\pi/4})(z-e^{-i\pi/4})(z-e^{i3\pi/4})(z-e^{-i3\pi/4})}$$

$$\operatorname{res} f(e^{i\pi/4}) = \frac{1}{4\sqrt{2}}(1-i), \quad \operatorname{res} f(e^{i3\pi/4}) = -\frac{1}{4\sqrt{2}}(1+i)。$$

$$\text{所以 } \text{v.p.} \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2\pi i [\operatorname{res} f(e^{i\pi/4}) + \operatorname{res} f(e^{i3\pi/4})] = \frac{1}{\sqrt{2}} \pi。$$

$$\text{被积函数是偶函数, 所以 } \int_0^{\infty} f(x) dx = \frac{1}{2} \lim_{b \rightarrow \infty} 2 \int_0^b f(x) dx = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\int_{-b}^0 f(x) dx + \int_0^b f(x) dx \right] \\ = \frac{1}{2} \text{v.p.} \int_{-\infty}^{\infty} f(x) dx, \text{ 即 } \int_0^{\infty} f(x) dx \text{ 收敛, 同样的, } \int_{-\infty}^0 f(x) dx \text{ 也收敛, 所以 } \int_{-\infty}^{\infty} f(x) dx \text{ 收}$$

$$\text{敛, 且等于 } \text{v.p.} \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx。 \text{ 后面类似的讨论省略。}$$

$$(2) \text{ 令 } f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}, \text{ 则}$$

$$\operatorname{res} f(ai) = \frac{i}{2a(a^2-b^2)}, \quad \operatorname{res} f(bi) = -\frac{i}{2b(a^2-b^2)}。$$

$$\text{原积分} = \text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i [\text{res } f(ai) + \text{res } f(bi)] = \frac{\pi}{ab(a+b)}.$$

$$(3) \text{ 令 } f(z) = \frac{z^{2m}}{z^{2n}+1}, \text{ 它在上半平面的奇点是 } e^{i(k+\frac{1}{2})\frac{\pi}{n}} \quad (k=0,1,2,\cdots,n-1).$$

$$\begin{aligned} \text{res } f \left[e^{i(k+\frac{1}{2})\frac{\pi}{n}} \right] &= \lim_{z \rightarrow e^{i(k+\frac{1}{2})\frac{\pi}{n}}} \frac{z^{2m}}{(z^{2n}+1)}, \quad \lim_{z \rightarrow e^{i(k+\frac{1}{2})\frac{\pi}{n}}} \frac{z^{2m-2n+1}}{2n} \\ &= \frac{1}{2n} e^{ik(2m-2n+1)\frac{\pi}{n}} e^{i(2m-2n+1)\frac{\pi}{2n}} \end{aligned}$$

$$\text{原积分} = \text{v.p.} \int_{-\infty}^{\infty} \frac{x^{2m}}{x^{2n}+1} dx$$

$$\begin{aligned} &= 2\pi i \sum_{k=0}^{n-1} \text{res } f \left[e^{i(k+\frac{1}{2})\frac{\pi}{n}} \right] = \frac{\pi i}{n} e^{i(2m-2n+1)\frac{\pi}{2n}} \sum_{k=0}^{n-1} e^{ik(2m-2n+1)\frac{\pi}{n}} \\ &= \frac{\pi i}{n} e^{i(2m-2n+1)\frac{\pi}{2n}} \frac{e^{i(2m-2n+1)\pi} - 1}{e^{i(2m-2n+1)\frac{\pi}{n}} - 1} = -\frac{\pi i}{2i \sin \frac{(2m-2n+1)\pi}{2n}} \\ &= \frac{\pi}{n \sin \frac{(2m+1)\pi}{2n}}. \end{aligned}$$

$$(4) \text{ 令 } f(z) = \frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z+i)^{n+1}(z-i)^{n+1}}, \text{ 则}$$

$$\text{res } f(i) = \frac{1}{n!} \left[\frac{1}{(z+i)^{n+1}} \right]_{z=i}^{(n)} = \frac{1}{n!} \frac{(-1)^n (n+1)(n+2)\cdots(2n)}{(2i)^{2n+1}} = \frac{1}{2i} \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{2i} \frac{(2n-1)!!}{(2n)!!}$$

$$\text{所以原积分} = \text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = 2\pi i \text{res } f(i) = \frac{(2n-1)!!}{(2n)!!}.$$

$$(5) \text{ 令 } f(z) = \frac{z^2}{(z^2+a^2)^2} = \frac{z^2}{(z+ai)^2(z-ai)^2}, \text{ 则 } \text{res } f(ai) = \lim_{z \rightarrow ai} \frac{d}{dz} \frac{z^2}{(z+ai)^2} = \frac{1}{4ai}.$$

$$\text{同第 (1) 小题的讨论, 有原积分} = \frac{1}{2} \text{v.p.} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)^2} = \frac{1}{2} \cdot 2\pi i \text{res } f(ai) = \frac{\pi}{4a}. \quad (\text{后面}$$

类似讨论省略)

$$(6) \frac{1}{x^2-2x+4} = O\left(\frac{1}{x^2}\right), \text{ 所以 } \int_0^{\infty} \frac{dx}{x^2-2x+4} \text{ 收敛, } \int_{-\infty}^0 \frac{dx}{x^2-2x+4} \text{ 也收敛, 所以}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+4} \text{ 收敛, 可直接计算其主值. } ((7) \text{ 可同样讨论}).$$

$$\text{令 } f(z) = \frac{1}{z^2-2z+4} = \frac{1}{(z-1-i\sqrt{3})(z-1+i\sqrt{3})}, \quad \text{res } f(1+i\sqrt{3}) = \frac{1}{2\sqrt{3}i},$$

$$\text{原积分} = \frac{\pi}{\sqrt{3}}.$$

$$(7) \text{ 令 } f(z) = \frac{z^2}{(z^2+1)(z^2-2z \cos \theta + 1)} = \frac{z^2}{(z+i)(z-i)(z-e^{i\theta})(z-e^{-i\theta})},$$

$$\text{res } f(i) = -\frac{1}{4 \cos \theta}, \quad \text{res } f(e^{i\theta}) = \frac{1}{2i \sin \theta (1+e^{-2i\theta})} = \frac{1}{4 \cos \theta} - \frac{i}{4 \sin \theta},$$

$$\text{res } f(e^{-i\theta}) = -\frac{e^{-2i\theta}}{2i \sin \theta (1+e^{-2i\theta})} = \frac{1}{4 \cos \theta} + \frac{i}{4 \sin \theta}.$$

$$\text{若 } \sin \theta > 0, \text{ 则 } e^{i\theta} \text{ 在上半平面, 原积分} = 2\pi i [\text{res } f(i) + \text{res } f(e^{i\theta})] = \frac{\pi}{2 \sin \theta}.$$

$$\text{若 } \sin \theta < 0, \text{ 则 } e^{-i\theta} \text{ 在上半平面, 原积分} = 2\pi i [\text{res } f(i) + \text{res } f(e^{-i\theta})] = -\frac{\pi}{2 \sin \theta}.$$

$$\text{所以原积分} = \frac{\pi}{2 |\sin \theta|}.$$

$$(8) \text{ 令 } f(z) = \frac{1}{(1+z^2) \text{ch } \frac{\pi z}{2}}, \text{ 则 } z=i \text{ 为二阶极点, } z=(2k+1)i \quad (k=1,2,\cdots) \text{ 为一阶}$$

$$\text{极点. } \text{res } f(i) = \lim_{z \rightarrow i} \frac{d}{dz} \frac{z-i}{(z+i) \text{ch } \frac{\pi z}{2}} = \lim_{z \rightarrow i} \frac{2i \text{ch } \frac{\pi z}{2} - \frac{\pi}{2} (z^2+1) \text{sh } \frac{\pi z}{2}}{(z+i)^2 \text{ch}^2 \frac{\pi z}{2}}$$

$$= \lim_{z \rightarrow i} \frac{\pi i \text{sh } \frac{\pi z}{2} - \pi z \text{sh } \frac{\pi z}{2} - \left(\frac{\pi}{2}\right)^2 (z^2+1) \text{ch } \frac{\pi z}{2}}{2(z+i) \text{ch}^2 \frac{\pi z}{2} + \pi (z+i)^2 \text{ch } \frac{\pi z}{2} \text{sh } \frac{\pi z}{2}}$$

$$= \lim_{z \rightarrow i} \frac{-\pi \text{sh } \frac{\pi z}{2} - \left(\frac{\pi}{2}\right)^2 (z+i) \text{ch } \frac{\pi z}{2}}{2(z+i) \text{ch } \frac{\pi z}{2} \frac{\text{ch } \frac{\pi z}{2}}{z-i} + \pi (z+i)^2 \text{sh } \frac{\pi z}{2} \frac{\text{ch } \frac{\pi z}{2}}{z-i}}$$

$$= \frac{-\pi \text{sh } \frac{\pi z}{2} - \left(\frac{\pi}{2}\right)^2 (z+i) \text{ch } \frac{\pi z}{2}}{\pi (z+i) \text{ch } \frac{\pi z}{2} \text{sh } \frac{\pi z}{2} + \frac{\pi^2}{2} (z+i)^2 \text{sh}^2 \frac{\pi z}{2}} \bigg|_{z=i} = \frac{1}{2\pi i}.$$

$$\text{res } f[(2k+1)i] = \lim_{z \rightarrow (2k+1)i} \frac{2}{\pi (z^2+1) \text{sh } \frac{\pi z}{2}} = \frac{(-1)^{k-1}}{2\pi i} \frac{1}{k(k+1)}. \quad (k=1,2,\cdots)$$

取这样的积分路径: 实轴上 $-R_n$ 到 R_n , 以原点为圆心, R_n 为半径的上半圆 C_n , 这里

$$R_n = 2n. \text{ 该闭合路径包围奇点 } (2k+1)i \quad (k=0,1,\cdots,n-1), \text{ 且路径上没有奇点. 有}$$

$$\int_{-2n}^{2n} f(x) dx + \int_{C_n} f(z) dz = 1 + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k(k+1)} = 1 + \sum_{k=1}^{n-1} (-1)^{k-1} \left(\frac{1}{k} - \frac{1}{k+1} \right). \quad (*)$$

$$\text{在 } C_n \text{ 上有 } z_n = 2n(\cos \theta + i \sin \theta), \quad \text{ch } \frac{\pi z_n}{2} = \frac{1}{2} [e^{n\pi(\cos \theta + i \sin \theta)} + e^{-n\pi(\cos \theta + i \sin \theta)}] \quad (0 \leq \theta \leq \pi),$$

$$\text{当 } 0 \leq \theta < \frac{\pi}{2} \text{ 时, } \cos \theta > 0, \quad e^{n\pi(\cos \theta + i \sin \theta)} \rightarrow \infty, e^{-n\pi(\cos \theta + i \sin \theta)} \rightarrow 0, \quad \text{ch } \frac{\pi z_n}{2} \rightarrow \infty,$$

$$\text{当 } \frac{\pi}{2} < \theta \leq \pi \text{ 时, } \cos \theta < 0, \quad e^{n\pi(\cos \theta + i \sin \theta)} \rightarrow 0, e^{-n\pi(\cos \theta + i \sin \theta)} \rightarrow \infty, \quad \text{ch } \frac{\pi z_n}{2} \rightarrow \infty,$$

$$\text{当 } \theta = \frac{\pi}{2} \text{ 时, } \text{ch } \frac{\pi z_n}{2} = \cos n\pi, \quad \left| \text{ch } \frac{\pi z_n}{2} \right| = 1.$$

综上, $0 \leq \theta \leq \pi$ 时存在 N , 当 $n > N$ 时有 $\left| \text{ch } \frac{\pi z_n}{2} \right| \geq 1$. 在 C_n 上

$$\left| \frac{1}{(1+z_n^2) \text{ch } \frac{\pi z_n}{2}} \right| \leq \frac{z_n}{1+z_n^2} \quad (n > N), \text{ 所以 } \lim_{n \rightarrow \infty} z_n f(z_n) = 0, \text{ 由此得 } \lim_{n \rightarrow \infty} \int_{C_n} f(z) dz = 0.$$

令 $(*)$ 式 $n \rightarrow \infty$ 可得

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2) \text{ch } \frac{\pi x}{2}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

$$\text{易知 } \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k, \quad (|x| < 1) \text{ 由于右边级数在 } x=1 \text{ 收敛, 根据 Abel 第二定理,}$$

$$\text{该级数在 } x=1 \text{ 点左连续, 对上式两边取 } x \rightarrow 1-0 \text{ 可得 } \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2.$$

$$\text{所以 } \text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2) \text{ch } \frac{\pi x}{2}} = 2 \ln 2.$$

$$\text{由于 } \text{ch } \frac{\pi x}{2} \geq 1, \text{ 所以 } \left| \frac{1}{(1+x^2) \text{ch } \frac{\pi x}{2}} \right| \leq \frac{1}{1+x^2}, \text{ 因为 } \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \text{ 收敛, 所以原积分收敛,}$$

且等于其主值。

$$38. \text{ 计算下面积分: (1) } \int_0^{\infty} \frac{\cos x}{1+x^4} dx; (2) \int_0^{\infty} \frac{x \sin mx}{x^2+a^2} dx, \quad a>0, m>0; \quad (3) \int_{-\infty}^{\infty} \frac{x \sin x}{x^2-2x+2} dx; (4) \int_0^{\infty} \frac{x^3 \sin mx}{x^4+4a^4} dx, \quad a>0, m>0; (5) \int_{-\infty}^{\infty} \frac{\cos mx}{(x+b)^2+a^2} dx,$$

$$\int_{-\infty}^{\infty} \frac{\sin mx}{(x+b)^2+a^2} dx, \quad a>0, m>0; (6) \int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2+a^2} dx, \quad a>0;$$

$$(7) \int_{-\infty}^{\infty} \frac{\sin^2 ax}{(x^2+b^2)(x^2+c^2)} dx, \quad a>0, b>0, c>0; (8) \int_0^{\infty} \frac{\cos x}{\text{ch } x} dx.$$

$$(1) \text{ 令 } f(z) = \frac{e^{iz}}{1+z^4} = \frac{e^{iz}}{(z-e^{i\pi/4})(z-e^{-i\pi/4})(z-e^{i3\pi/4})(z-e^{-i3\pi/4})}.$$

$$\text{res } f(e^{i\pi/4}) = \frac{e^{-1/\sqrt{2}}}{4i} e^{i\left(\frac{1}{\sqrt{2}} \frac{\pi}{4}\right)}, \quad \text{res } f(e^{i3\pi/4}) = \frac{e^{-1/\sqrt{2}}}{4i} e^{-i\left(\frac{1}{\sqrt{2}} \frac{\pi}{4}\right)}.$$

$$\text{原积分} = \frac{1}{2} \text{v.p.} \int_{-\infty}^{\infty} \frac{\cos x}{1+x^4} dx$$

$$= \frac{1}{2} \text{Re} \left\{ 2\pi i [\text{res } f(e^{i\pi/4}) + \text{res } f(e^{i3\pi/4})] \right\} = \frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}} \cos \left(\frac{1}{\sqrt{2}} - \frac{\pi}{4} \right).$$

$$(2) \text{ 令 } f(z) = \frac{ze^{imz}}{z^2+a^2} = \frac{ze^{imz}}{(z+ai)(z-ai)}, \text{ 则 } \text{res } f(ai) = \frac{1}{2} e^{-ma}.$$

$$\text{原积分} = \frac{1}{2} \text{v.p.} \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2+a^2} dx = \frac{1}{2} \text{Im} [2\pi i \text{res } f(ai)] = \frac{\pi}{2} e^{-ma}.$$

$$(3) \text{ 令 } f(z) = \frac{ze^{iz}}{z^2-2z+2} = \frac{ze^{iz}}{(z-1-i)(z-1+i)}, \text{ 则 } \text{res } f(1+i) = \frac{1}{\sqrt{2}ei} e^{i\left(1+\frac{\pi}{4}\right)}.$$

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2-2x+2} dx = \text{Im} [2\pi i \cdot \text{res } f(1+i)] = \frac{\sqrt{2}\pi}{e} \sin \left(1 + \frac{\pi}{4} \right).$$

由于 $\frac{x}{x^3-2x+2}$ 单调趋于 0, $\left|\int_0^b \sin x dx\right| \leq 2$, 所以原积分收敛 (狄里克莱判敛法), 等于其主值。

$$(4) f(z) = \frac{z^3 e^{imz}}{z^4 + 4a^4} = \frac{z^3 e^{imz}}{\left(z - \sqrt[4]{2}ae^{i\frac{\pi}{4}}\right)\left(z - \sqrt[4]{2}ae^{-i\frac{\pi}{4}}\right)\left(z - \sqrt[4]{2}ae^{i\frac{3\pi}{4}}\right)\left(z - \sqrt[4]{2}ae^{-i\frac{3\pi}{4}}\right)}.$$

$$\text{则 } \operatorname{res} f\left(\sqrt[4]{2}ae^{i\pi/4}\right) = \frac{1}{4}e^{-ma}e^{-ima}, \quad \operatorname{res} f\left(\sqrt[4]{2}ae^{i3\pi/4}\right) = \frac{1}{4}e^{-ma}e^{-ima}.$$

$$\begin{aligned} \text{原积分} &= \frac{1}{2} \operatorname{v.p.} \int_0^\infty \frac{x^3 \sin mx}{x^4 + 4a^4} dx \\ &= \frac{1}{2} \operatorname{Im} \left\{ 2\pi i \left[\operatorname{res} f\left(\sqrt[4]{2}ae^{i\pi/4}\right) + \operatorname{res} f\left(\sqrt[4]{2}ae^{i3\pi/4}\right) \right] \right\} = \frac{\pi}{2} e^{-ma} \cos ma. \end{aligned}$$

$$(5) \text{ 令 } f(z) = \frac{e^{imz}}{(z+b)^2 + a^2} = \frac{e^{imz}}{(z+b+ai)(z+b-ai)}, \quad \operatorname{res} f(-b+ai) = \frac{e^{-ma}}{2ai} e^{-imb}.$$

$$\text{所以 } \operatorname{v.p.} \int_{-\infty}^\infty \frac{\cos mx}{(x+b)^2 + a^2} dx = \operatorname{Re} [2\pi i \operatorname{res} f(-b+ai)] = \frac{\pi}{a} e^{-ma} \cos mb.$$

$$\operatorname{v.p.} \int_{-\infty}^\infty \frac{\sin mx}{(x+b)^2 + a^2} dx = \operatorname{Im} [2\pi i \operatorname{res} f(-b+ai)] = -\frac{\pi}{a} e^{-ma} \sin mb.$$

由于 $\left| \frac{\cos x}{(x+b)^2 + a^2} \right| \leq \frac{1}{(x+b)^2 + a^2}$, $\left| \frac{\sin x}{(x+b)^2 + a^2} \right| \leq \frac{1}{(x+b)^2 + a^2}$ 所以原积分收敛, 等于其主值。

$$(6) \text{ 令上小题第一个积分中 } b=0, m=1 \text{ 可得 } \int_{-\infty}^\infty \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a}.$$

$$\text{令第 (2) 小题中 } m=1 \text{ 可得 } \operatorname{v.p.} \int_{-\infty}^\infty \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}.$$

所以原积分 $= 2\pi e^{-a}$ 。

$$(7) \text{ 原积分} = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{(x^2 + b^2)(x^2 + c^2)} dx - \frac{1}{2} \int_{-\infty}^\infty \frac{\cos 2ax}{(x^2 + b^2)(x^2 + c^2)} dx.$$

$$\text{令 } f(z) = \frac{1}{(z^2 + b^2)(z^2 + c^2)} = \frac{1}{(z+bi)(z-bi)(z+ci)(z-ci)},$$

$$\operatorname{res} f(bi) = -\frac{1}{2b(b^2 - c^2)i}, \quad \operatorname{res} f(ci) = \frac{1}{2c(b^2 - c^2)i}.$$

$$\text{则 } \int_{-\infty}^\infty \frac{1}{(x^2 + b^2)(x^2 + c^2)} dx = 2\pi i [\operatorname{res} f(bi) + \operatorname{res} f(ci)] = \frac{\pi}{bc(b+c)}.$$

$$\text{令 } f(z) = \frac{e^{2iaz}}{(z^2 + b^2)(z^2 + c^2)} = \frac{e^{2iaz}}{(z+bi)(z-bi)(z+ci)(z-ci)},$$

$$\operatorname{res} f(bi) = -\frac{e^{-2ab}}{2b(b^2 - c^2)i}, \quad \operatorname{res} f(ci) = \frac{e^{-2ac}}{2c(b^2 - c^2)i}. \text{ 则}$$

$$\operatorname{v.p.} \int_{-\infty}^\infty \frac{\cos 2ax}{(x^2 + b^2)(x^2 + c^2)} dx = \operatorname{Re} \{ 2\pi i [\operatorname{res} f(bi) + \operatorname{res} f(ci)] \} = \frac{\pi}{b^2 - c^2} \left(\frac{e^{-2ac}}{c} - \frac{e^{-2ab}}{b} \right)$$

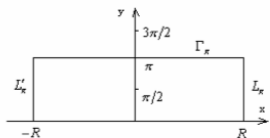
由于 $\left| \frac{\sin^2 ax}{(x^2 + b^2)(x^2 + c^2)} \right| \leq \frac{1}{(x^2 + b^2)(x^2 + c^2)}$, 所以原积分收敛且

$$\text{原积分} = \frac{\pi(b-c+ce^{-2ab}-be^{-2ac})}{2bc(b^2 - c^2)}.$$

$$(8) \text{ 令 } f(z) = \frac{e^{iz}}{\operatorname{ch} z}, \text{ 它有一阶极点 } \left(k + \frac{1}{2}\right)\pi i \quad (k=0,1,\dots).$$

$$\operatorname{res} f\left[\left(k + \frac{1}{2}\right)\pi i\right] = \lim_{z \rightarrow \left(k + \frac{1}{2}\right)\pi i} \frac{e^{iz}}{\operatorname{sh} z} = \frac{(-1)^k}{i} e^{-\left(k + \frac{1}{2}\right)\pi}, \quad \operatorname{res} f\left(\frac{\pi}{2}i\right) = \frac{1}{i} e^{-\frac{\pi}{2}}.$$

选取如下积分路径:



$$\int_{-R}^R f(x) dx + \int_{L_R} f(z) dz + \int_{\Gamma_R} f(z) dz + \int_{L_\pi} f(z) dz = 2\pi i \operatorname{res} f\left(\frac{\pi}{2}i\right) = 2\pi e^{-\frac{\pi}{2}}. \quad (*)$$

$$\text{因为 } |\operatorname{ch}(R + yi)| = |\operatorname{ch} R \cos y + i \operatorname{sh} R \sin y| = \sqrt{\operatorname{ch}^2 R \cos^2 y + \operatorname{sh}^2 R \sin^2 y}$$

$$= \sqrt{\cos^2 y + \operatorname{sh}^2 R} \geq \operatorname{sh} R, \text{ 所以 } \left| \int_{L_R} f(z) dz \right| = \left| \int_0^\pi \frac{e^{i(R+yi)}}{\operatorname{ch}(R+yi)} dy \right| \leq \int_0^\pi \frac{e^{-y}}{\operatorname{sh} R} dy = \frac{2(1-e^{-\pi})}{e^R - e^{-R}}.$$

$$\text{因此 } \lim_{R \rightarrow \infty} \int_{L_R} f(z) dz = 0, \text{ 同样有 } \lim_{R \rightarrow \infty} \int_{L_\pi} f(z) dz = 0.$$

$$\int_{\Gamma_R} f(z) dz = \int_R^{-R} \frac{e^{i(x+\pi i)}}{\operatorname{ch}(x+\pi i)} dx = -e^{-\pi} \int_R^{-R} \frac{e^{ix}}{\operatorname{ch} x} dx = e^{-\pi} \int_{-R}^R f(x) dx, \text{ 代入 } (*) \text{ 式并令}$$

$$R \rightarrow \infty \text{ 得 } \operatorname{v.p.} \int_{-\infty}^\infty \frac{e^{ix}}{\operatorname{ch} x} dx = \frac{\pi e^{-\pi/2}}{1 + e^{-\pi}} = \frac{\pi}{\operatorname{ch} \frac{\pi}{2}}. \text{ 所以原积分} = \frac{\pi}{2 \operatorname{ch} \frac{\pi}{2}}.$$

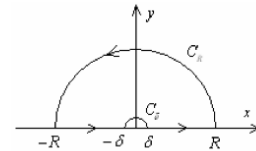
$$\text{计算下列积分: (1) } \int_0^\infty \frac{\sin mx}{x(x^2 + a^2)} dx, \quad a > 0, m > 0; (2) \operatorname{v.p.} \int_{-\infty}^\infty \frac{dx}{x(x-1)(x-2)};$$

$$(3) \int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx, \quad a > 0, b > 0; (4) \int_0^\infty \frac{\sin(x+a)\sin(x-a)}{x^2 - a^2} dx, \quad a > 0;$$

$$(5) \int_0^\infty \frac{\sin^3 x}{x^3} dx; \quad (6) \int_{-\infty}^\infty \frac{e^{px} - e^{qx}}{1 - e^x} dx, \quad 0 < p < 1, 0 < q < 1;$$

$$(7) \operatorname{v.p.} \int_{-\infty}^\infty \frac{x \cos x}{x^2 - 5x - 6} dx; \quad (8) \operatorname{v.p.} \int_{-\infty}^\infty \frac{\sin x}{(x^2 + 4)(x-1)} dx.$$

(1) 取下图积分路径:



$$\text{令 } f(z) = \frac{e^{imz}}{z(z^2 + a^2)}, \text{ 则 } \operatorname{res} f(ai) = -\frac{e^{-ma}}{2a^2}, \text{ 所以}$$

$$\begin{aligned} \int_{-R}^{-\delta} \frac{e^{imz}}{x(x^2 + a^2)} dx + \int_{C_\delta} \frac{e^{imz}}{z(z^2 + a^2)} dz + \int_\delta^R \frac{e^{imz}}{x(x^2 + a^2)} dx + \int_{C_R} \frac{e^{imz}}{z(z^2 + a^2)} dz \\ = 2\pi i \operatorname{res} f(ai) = -\frac{\pi i e^{-ma}}{a^2}. \quad (*) \end{aligned}$$

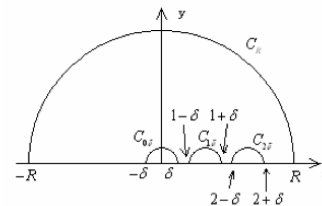
$$\lim_{\delta \rightarrow 0} \int_{C_\delta} \frac{e^{imz}}{z(z^2 + a^2)} dz = -\pi i \lim_{z \rightarrow 0} z \cdot \frac{e^{imz}}{z(z^2 + a^2)} = -\frac{\pi i}{a^2},$$

$$\text{因为 } \lim_{z \rightarrow \infty} \frac{e^{imz}}{z(z^2 + a^2)} = 0, \text{ 所以 } \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{imz}}{z(z^2 + a^2)} dz = 0,$$

$$\text{令 } (*) \text{ 式 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得到 } \operatorname{v.p.} \int_{-\infty}^\infty \frac{e^{mx}}{x(x^2 + a^2)} dx = \frac{\pi i}{a^2} (1 - e^{-ma}),$$

$$\text{所以 } \int_0^\infty \frac{\sin mx}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} (1 - e^{-ma}) = \frac{\pi}{a^2} e^{-\frac{1}{2}ma} \operatorname{sh} \frac{ma}{2}.$$

(2) 取下图积分路径:



$$\text{令 } f(z) = \frac{1}{z(z-1)(z-2)}, \text{ 则 } \lim_{\delta \rightarrow 0} \int_{C_{\delta\delta}} f(z) dz = -\pi i \lim_{z \rightarrow 0} \frac{1}{(z-1)(z-2)} = -\frac{\pi}{2} i,$$

$$\lim_{\delta \rightarrow 0} \int_{C_{1\delta}} f(z) dz = -\pi i \lim_{z \rightarrow 1} \frac{1}{z(z-2)} = \pi i,$$

$$\lim_{\delta \rightarrow 0} \int_{C_{2\delta}} f(z) dz = -\pi i \lim_{z \rightarrow 2} \frac{1}{z(z-1)} = -\frac{\pi}{2} i$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = \pi i \lim_{z \rightarrow \infty} \frac{1}{(z-1)(z-2)} = 0.$$

$$\int_{-R}^{-\delta} f(x) dx + \int_{C_{\delta\delta}} f(z) dz + \int_\delta^{1-\delta} f(x) dx + \int_{C_{1\delta}} f(z) dz + \int_{1+\delta}^{2-\delta} f(x) dx + \int_{C_{2\delta}} f(z) dz$$

$$+ \int_{2+\delta}^R f(x) dx + \int_{C_R} f(z) dz = 0$$

$$\text{令上式 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得到 } \operatorname{v.p.} \int_{-\infty}^\infty \frac{dx}{x(x-1)(x-2)} = 0.$$

$$(3) \text{ 令 } f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}, \text{ 取与第 (1) 小题相同的积分路径,}$$

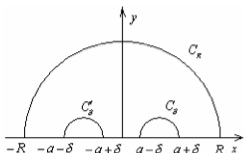
$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow 0} \frac{e^{iaz} - e^{ibz}}{z} = \pi(a-b),$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iaz}}{z^2} dz - \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{ibz}}{z^2} dz = 0,$$

对该路径的积分取 $\delta \rightarrow 0, R \rightarrow \infty$ 的极限可得 $\int_0^{\infty} \frac{e^{ax} \cos bx - e^{bx} \cos ax}{x^2} dx = \frac{\pi}{2} (b-a)$ 。

$$(4) \text{ 原积分} = \frac{1}{2} \int_0^{\infty} \frac{\cos 2a - \cos 2x}{x^2 - a^2} dx. \text{ 令 } f(z) = \frac{\cos 2a - e^{2iz}}{z^2 - a^2},$$

取如下积分路径:



$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow a} f(z) = \frac{\pi(\cos 2a - e^{2ia})}{2ai},$$

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow -a} f(z) = -\frac{\pi(\cos 2a - e^{-2ia})}{2ai}.$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

$$\int_{-R}^{-a-\delta} f(x) dx + \int_{C_\delta} f(z) dz + \int_{-a+\delta}^{a-\delta} f(x) dx + \int_{C_\delta} f(z) dz + \int_{a+\delta}^R f(x) dx + \int_{C_R} f(z) dz = 0$$

$$\text{令 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } v.p. \int_{-\infty}^{\infty} \frac{\cos 2a - e^{2ix}}{x^2 - a^2} dx = \frac{\pi}{a} (\sin 2a - i \cos 2a).$$

$$\text{所以原积分} = \frac{1}{4} \operatorname{Re} \left(v.p. \int_{-\infty}^{\infty} \frac{\cos 2a - e^{2ix}}{x^2 - a^2} dx \right) = \frac{\pi}{4a} \sin 2a.$$

$$(5) \sin^3 x = \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) \sin x = \frac{1}{2} \sin x - \frac{1}{4} (\sin 3x - \sin x) = \frac{1}{4} (3 \sin x - \sin 3x).$$

$$\text{原积分} = \int_0^{\infty} \frac{3 \sin x - \sin 3x}{4x^3} dx. \text{ 令 } f(z) = \frac{3e^{iz} - e^{3iz}}{4z^3}, \text{ 取与第(1)小题相同的积分}$$

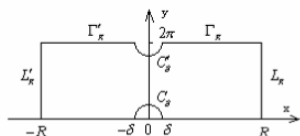
路径, 有

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow 0} \frac{3e^{iz} - e^{3iz}}{4z^2} = -\frac{3}{4} \pi i, \quad \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

$$\text{令围线积分 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } v.p. \int_{-\infty}^{\infty} \frac{3e^{ix} - e^{3ix}}{4x^3} dx = \frac{3}{4} \pi i,$$

$$\text{原积分} = \frac{1}{2} \operatorname{Im} \left(v.p. \int_{-\infty}^{\infty} \frac{3e^{ix} - e^{3ix}}{4x^3} dx \right) = \frac{3}{8} \pi.$$

$$(6) \text{ 令 } f(z) = \frac{e^{pz}}{1 - e^z}, \quad 2k\pi i \quad (k=0, \pm 1, \pm 2, \dots) \text{ 是其一阶极点. 取如下积分路径:}$$



$$\int_{-R}^{-\delta} f(x) dx + \int_{C_\delta} f(z) dz + \int_{\delta}^R f(x) dx + \int_{L_R} f(z) dz + \int_{\Gamma_R} f(z) dz + \int_{C_\delta} f(z) dz + \int_{\Gamma_R^*} f(z) dz + \int_{L_R^*} f(z) dz = 0.$$

$$\int_{\Gamma_R} f(z) dz = \int_{\delta}^R \frac{e^{p(x+2\pi i)}}{1 - e^{(x+2\pi i)}} dx = -e^{2p\pi i} \int_{\delta}^R f(x) dx, \quad \int_{\Gamma_R^*} f(z) dz = -e^{2p\pi i} \int_{-R}^{-\delta} f(x) dx,$$

代入 (*) 式得

$$(1 - e^{2p\pi i}) \left[\int_{-R}^{-\delta} f(x) dx + \int_{\delta}^R f(x) dx \right] + \int_{C_\delta} f(z) dz + \int_{L_R} f(z) dz + \int_{C_\delta} f(z) dz + \int_{L_R^*} f(z) dz = 0.$$

$$\left| \int_{\Gamma_R} f(z) dz \right| \leq \int_0^{2\pi} \left| \frac{e^{p(R+iy)}}{1 - e^{(R+iy)}} \right| dy = \int_0^{2\pi} \frac{e^{pR}}{\sqrt{(1 - e^R \cos y)^2 + e^{2R} \sin^2 y}} dy = e^{pR} \int_0^{2\pi} \frac{1}{\sqrt{1 - 2e^R \cos y + e^{2R}}} dy \leq e^{pR} \int_0^{2\pi} \frac{1}{e^R - 1} dy = 2\pi \frac{e^{(1-p)R}}{1 - e^{-R}},$$

$$\text{所以 } \lim_{R \rightarrow \infty} \int_{\Gamma_R} f(z) dz = 0, \quad \text{同样的, } \lim_{R \rightarrow \infty} \int_{\Gamma_R^*} f(z) dz = 0.$$

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow 0} \frac{ze^{pz}}{1 - e^z} = \pi i,$$

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow 2\pi i} \frac{(z - 2\pi i)e^{pz}}{1 - e^z} = \pi i e^{2p\pi i},$$

$$\text{令 (**) 式 } \delta \rightarrow 0, R \rightarrow \infty \text{ 可得 } v.p. \int_{-\infty}^{\infty} \frac{e^{px}}{1 - e^x} dx = \pi i \frac{e^{2p\pi i} + 1}{e^{2p\pi i} - 1} = \pi \cot(p\pi).$$

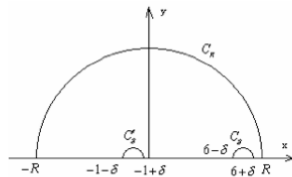
$$\text{所以 } v.p. \int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 - e^x} dx = v.p. \int_{-\infty}^{\infty} \frac{e^{px}}{1 - e^x} dx - v.p. \int_{-\infty}^{\infty} \frac{e^{qx}}{1 - e^x} dx = \pi [\cot(p\pi) - \cot(q\pi)].$$

当 x 充分大时, 有 $e^x - 1 \geq \frac{1}{2} e^x$ (即 $e^x \geq 2$). 不妨假设 $p > q$,

$$\left| \frac{e^{px} - e^{qx}}{1 - e^x} \right| \leq 2 \frac{e^{px} - e^{qx}}{e^x} = 2 [e^{-(1-p)x} - e^{-(1-q)x}]. \text{ 由于 } \int_0^{\infty} [e^{-(1-p)x} - e^{-(1-q)x}] dx \text{ 收敛, 所以}$$

$$\int_0^{\infty} \frac{e^{px} - e^{qx}}{1 - e^x} dx \text{ 收敛, 类似地, } \int_{-\infty}^0 \frac{e^{px} - e^{qx}}{1 - e^x} dx \text{ 收敛, 所以原积分收敛, 等于其主值.}$$

$$(7) \text{ 令 } f(z) = \frac{ze^{iz}}{z^2 - 5z - 6} = \frac{ze^{iz}}{(z-6)(z+1)}. \text{ 取如下积分路径:}$$



$$\left(\int_{-R}^{-1-\delta} + \int_{1+\delta}^R \right) f(x) dx + \left(\int_{C_\delta} + \int_{C_R} + \int_{C_R} \right) f(z) dz = 0.$$

$$\text{因为 } \lim_{z \rightarrow \infty} \frac{z}{(z-6)(z+1)} = 0, \text{ 所以 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

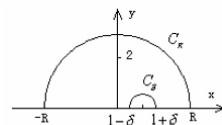
$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow -1} \frac{ze^{iz}}{z+1} = \frac{6\pi}{7i} e^{\delta i}, \quad \lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow -1} \frac{ze^{iz}}{z-6} = \frac{\pi}{7i} e^{-i}.$$

$$\text{令 (*) 式 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } v.p. \int_{-\infty}^{\infty} f(x) dx = \frac{\pi i}{7} (6e^{6i} + e^{-i}),$$

$$\text{原积分} = \operatorname{Re} \left[v.p. \int_{-\infty}^{\infty} f(x) dx \right] = \frac{\pi}{7} (\sin 1 - 6 \sin 6).$$

$$(8) \text{ 令 } f(z) = \frac{e^{iz}}{(z^2 + 4)(z - 1)}, \quad \operatorname{res} f(2i) = \lim_{z \rightarrow 2i} \frac{e^{iz}}{(z + 2i)(z - 1)} = -\frac{e^{-2}}{20i} (1 + 2i).$$

取如下积分路径:



$$\left(\int_{-R}^{1-\delta} + \int_{1+\delta}^R \right) f(x) dx + \left(\int_{C_\delta} + \int_{C_R} \right) f(z) dz = 2\pi i \operatorname{res} f(2i) = -\frac{\pi e^{-2}}{10} (1 + 2i).$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0, \quad \lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -\pi i \lim_{z \rightarrow 1} \frac{e^{iz}}{z^2 + 4} = -\frac{\pi i}{5} e^i.$$

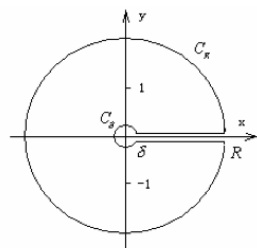
$$\text{令 (*) } \delta \rightarrow 0, R \rightarrow \infty \text{ 得原积分} = \operatorname{Im} \left[v.p. \int_{-\infty}^{\infty} f(x) dx \right] = \frac{\pi}{5} (\cos 1 - e^{-2}).$$

$$40. \text{ 计算下列积分: (1) } \int_0^{\infty} \frac{x^s}{(1+x^2)^2} dx, \quad -1 < s < 3; (2) \int_0^{\infty} \frac{x^{-p}}{1+2x \cos \lambda + x^2} dx,$$

$$-1 < p < 1, \quad 0 < \lambda < \pi; (3) v.p. \int_0^{\infty} \frac{x^{a-1}}{1-x} dx, \quad 0 < a < 1; (4) \int_0^{\infty} \frac{\ln x}{\sqrt{x}(x^2 + a^2)^2} dx, \quad a > 0.$$

$$(1) \text{ 令 } f(z) = \frac{z^s}{(1+z^2)^2}, \text{ 取如下积分路径, 规定 } 0 \leq \arg z \leq 2\pi,$$

(8.32)



$$\text{则 } \operatorname{res} f(i) = \lim_{z \rightarrow i} \frac{d}{dz} \frac{z^s}{(z+i)^2} = -\frac{s-1}{4i} e^{\frac{i\pi}{2}}, \quad \operatorname{res} f(-i) = \lim_{z \rightarrow -i} \frac{d}{dz} \frac{z^s}{(z-i)^2} = \frac{s-1}{4i} e^{\frac{3i\pi}{2}}$$

$$(\text{这里 } i = e^{\frac{i\pi}{2}}, \quad -i = e^{\frac{3i\pi}{2}}). \text{ 围道积分为:}$$

$$\int_{\delta}^R \frac{x^s}{(1+x^2)^2} dx + \int_R^{\delta} \frac{x^s e^{2i\pi}}{(1+x^2)^2} dx + \left(\int_{C_\delta} + \int_{C_R} \right) f(z) dz = 2\pi i [\operatorname{res} f(i) + \operatorname{res} f(-i)] = i\pi (s-1) e^{i\pi} \sin \frac{s\pi}{2}.$$

$$\text{即 } (1 - e^{2i\pi}) \int_{\delta}^R \frac{x^s}{(1+x^2)^2} dx + \left(\int_{C_\delta} + \int_{C_R} \right) f(z) dz = i\pi (s-1) e^{i\pi} \sin \frac{s\pi}{2}.$$

41. 令上式 $\delta \rightarrow 0, R \rightarrow \infty$, 由于 $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 2\pi i \lim_{z \rightarrow \infty} z \cdot \frac{z^s}{(1+z^2)^2} = 0$, 设 $P(z)$ 及 $Q(z)$ 分别为 m 阶及 n 阶多项式, 并且 $m \leq n-2$, 且 $Q(z)$ 无非负实根。

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -2\pi i \lim_{z \rightarrow 0} z \cdot \frac{z^s}{(1+z^2)^2} = 0, \text{ 所以}$$

$$\int_0^\infty \frac{x^s}{(1+x^2)^2} dx = \frac{i\pi(s-1)e^{i\pi} \sin \frac{s\pi}{2}}{1-e^{2i\pi}} = \frac{\pi}{4}(1-s) \sec \frac{s\pi}{2}.$$

$$(2) \text{ 令 } f(z) = \frac{z^{-p}}{z^2 + 2z \cos \lambda + 1} = \frac{z^{-p}}{(z + e^{i\lambda})(z + e^{-i\lambda})}, \text{ 取与上小题同样的积分路径, 规定}$$

$$0 \leq \arg z \leq 2\pi, \text{ 则 } \operatorname{res} f \left[e^{i(\pi+\lambda)} \right] = \lim_{z \rightarrow e^{i(\pi+\lambda)}} \frac{z^{-p}}{z + e^{-i\lambda}} = -\frac{e^{-ip(\pi+\lambda)}}{2i \sin \lambda},$$

$$\operatorname{res} f \left[e^{i(\pi-\lambda)} \right] = \lim_{z \rightarrow e^{i(\pi-\lambda)}} \frac{z^{-p}}{z + e^{i\lambda}} = \frac{e^{-ip(\pi-\lambda)}}{2i \sin \lambda}, \text{ 所以}$$

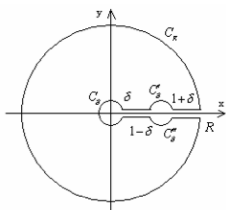
$$(1 - e^{-2ip\pi}) \int_\delta^R \frac{x^{-p}}{1 + 2x \cos \lambda + x^2} dx + \left(\int_{C_R} + \int_{C_\delta} \right) f(z) dz = 2\pi i e^{-ip\pi} \frac{\sin p\lambda}{\sin \lambda}.$$

$$\text{令上式 } \delta \rightarrow 0, R \rightarrow \infty, \text{ 由于 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 2\pi i \lim_{z \rightarrow \infty} \frac{z^{-p}}{z^2 + 2z \cos \lambda + 1} = 0,$$

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -2\pi i \lim_{z \rightarrow 0} z \cdot \frac{z^{-p}}{z^2 + 2z \cos \lambda + 1} = 0, \text{ 所以}$$

$$\int_0^\infty \frac{x^{-p}}{1 + 2x \cos \lambda + x^2} dx = 2\pi i e^{-ip\pi} \frac{\sin p\lambda}{\sin \lambda (1 - e^{-2ip\pi})} = \frac{\pi \sin p\lambda}{\sin p\pi \sin \lambda}.$$

$$(3) \text{ 令 } f(z) = \frac{z^{a-1}}{1-z}, \text{ 取如下积分路径, 规定 } 0 \leq \arg z \leq 2\pi.$$



$$(1 - e^{2ia\pi}) \left(\int_\delta^{1-\delta} + \int_{1+\delta}^R \right) f(x) dx + \left(\int_{C_R} + \int_{C_\delta} + \int_{C'_\delta} + \int_{C'_R} \right) f(z) dz = 0,$$

$$\text{令上式 } \delta \rightarrow 0, R \rightarrow \infty, \text{ 因为 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0, \lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = 0,$$

$$\lim_{\delta \rightarrow 0} \int_{C'_\delta} f(z) dz = -\pi i \lim_{z \rightarrow 1} (z-1) \frac{z^{a-1}}{1-z} = \pi i,$$

$$\lim_{\delta \rightarrow 0} \int_{C'_R} f(z) dz = -\pi i \lim_{z \rightarrow e^{2i\pi}} (z-1) \frac{z^{a-1}}{1-z} = \pi i e^{2ia\pi},$$

$$\text{所以原积分} = \pi i \frac{e^{2ia\pi} + 1}{e^{2ia\pi} - 1} = \pi \cot(a\pi).$$

$$(4) \text{ 令 } f(z) = \frac{(\ln z)^2}{\sqrt{z}(z^2 + a^2)}, \text{ 取与第(1)小题相同的积分路径, 规定 } 0 \leq \arg z \leq 2\pi,$$

$$\text{则 } \operatorname{res} f(ai) = \lim_{z \rightarrow ai} \frac{d}{dz} \frac{(\ln z)^2}{\sqrt{z}(z+ai)^2} = \lim_{z \rightarrow ae^{i\pi/2}} \frac{4(z+ai) \ln z - (5z+ai)(\ln z)^2}{2z^{3/2}(z+ai)^3}$$

$$= \frac{4 \ln a - 3(\ln a)^2 + \frac{3}{4}\pi^2 - 2\pi + 3\pi \ln a + i \left[4 \ln a - 3(\ln a)^2 + \frac{3}{4}\pi^2 + 2\pi - 3\pi \ln a \right]}{8\sqrt{2}a^{7/2}},$$

$$\operatorname{res} f(-ai) = \lim_{z \rightarrow -ai} \frac{d}{dz} \frac{(\ln z)^2}{\sqrt{z}(z-ai)^2} = \lim_{z \rightarrow ae^{3\pi/2}} \frac{4(z-ai) \ln z - (5z-ai)(\ln z)^2}{2z^{3/2}(z-ai)^3}$$

$$= \frac{-4 \ln a + 3(\ln a)^2 - \frac{27}{4}\pi^2 - 6\pi + 9\pi \ln a + i \left[4 \ln a - 3(\ln a)^2 + \frac{27}{4}\pi^2 - 6\pi + 9\pi \ln a \right]}{8\sqrt{2}a^{7/2}}.$$

围道积分为:

$$\int_\delta^R \frac{(\ln x)^2}{\sqrt{x}(x^2 + a^2)^2} dx + \int_R^\delta \frac{(\ln x + 2\pi i)^2}{-\sqrt{x}(x^2 + a^2)^2} dx + \left(\int_{C_R} + \int_{C_\delta} \right) f(z) dz = 2\pi i [\operatorname{res} f(ai) + \operatorname{res} f(-ai)]$$

$$\text{令 } \delta \rightarrow 0, R \rightarrow \infty, \text{ 由于 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0, \lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = 0, \text{ 所以}$$

$$\int_\delta^\infty \frac{2(\ln x)^2 - 4\pi^2}{\sqrt{x}(x^2 + a^2)^2} dx + 4\pi i \int_0^\infty \frac{\ln x}{\sqrt{x}(x^2 + a^2)^2} dx = 2\pi i [\operatorname{res} f(ai) + \operatorname{res} f(-ai)],$$

$$\text{因此原积分} = \frac{1}{4\pi} \operatorname{Im} \{ 2\pi i [\operatorname{res} f(ai) + \operatorname{res} f(-ai)] \} = \frac{1}{2} \operatorname{Re} [\operatorname{res} f(ai) + \operatorname{res} f(-ai)]$$

$$= \frac{\pi}{2\sqrt{2}a^{7/2}} \left(\frac{3}{2} \ln a - 1 - \frac{3\pi}{4} \right).$$

42. 利用上题结果计算下列积分: (1) $\int_0^\infty \frac{x}{(1+x+x^2)^2} dx$; (2) $\int_0^\infty \frac{1}{x^3+a^3} dx$; (3) $\int_0^\infty \frac{1}{(x+a)(x^2+b^2)} dx, a>0, b>0$; (4) $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx$.

考虑函数 $\frac{P(z)}{Q(z)} \ln z$ 的积分, 证明 $\int_0^\infty \frac{P(x)}{Q(x)} dx = -\sum_{\text{全平面}} \operatorname{res} \left\{ \frac{P(z)}{Q(z)} \ln z \right\}, 0 \leq \arg z \leq 2\pi$.

证: 令 $f(z) = \frac{P(z)}{Q(z)} \ln z$, 取与上题第(1)小题相同的积分路径, 规定 $0 \leq \arg z \leq 2\pi$,

$$\text{有 } \int_\delta^R \frac{P(x)}{Q(x)} \ln x dx + \int_R^\delta \frac{P(x)}{Q(x)} (\ln x + 2\pi i) dx + \left(\int_{C_R} + \int_{C_\delta} \right) f(z) dz = 2\pi i \sum_{\text{全平面}} \operatorname{res} \left\{ \frac{P(z)}{Q(z)} \ln z \right\},$$

$$\text{即 } \int_\delta^R \frac{P(x)}{Q(x)} dx + \left(\int_{C_R} + \int_{C_\delta} \right) f(z) dz = -\sum_{\text{全平面}} \operatorname{res} \left\{ \frac{P(z)}{Q(z)} \ln z \right\}. (*)$$

由于 0 不是 $Q(z)$ 的零点, 则 $\lim_{z \rightarrow 0} z \cdot \frac{P(z)}{Q(z)} \ln z = \frac{P(0)}{Q(0)} \lim_{z \rightarrow 0} z \ln z = 0$, 所以

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = 0. \text{ 因为 } m \leq n-2, \text{ 所以 } \lim_{z \rightarrow \infty} z \cdot \frac{P(z)}{Q(z)} \ln z = \lim_{z \rightarrow \infty} \frac{\ln z}{z^{n-m-1}} = 0, \text{ 则}$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0. \text{ 令 } (*) \text{ 式 } \delta \rightarrow 0, R \rightarrow \infty \text{ 即得 } \int_0^\infty \frac{P(x)}{Q(x)} dx = -\sum_{\text{全平面}} \operatorname{res} \left\{ \frac{P(z)}{Q(z)} \ln z \right\}.$$

42. 利用上题结果计算下列积分: (1) $\int_0^\infty \frac{x}{(1+x+x^2)^2} dx$; (2) $\int_0^\infty \frac{1}{x^3+a^3} dx$; (3) $\int_0^\infty \frac{1}{(x+a)(x^2+b^2)} dx, a>0, b>0$; (4) $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx$.

$$(3) \int_0^\infty \frac{1}{(x+a)(x^2+b^2)} dx, a>0, b>0; (4) \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)} dx.$$

$$(1) \text{ 令 } f(z) = \frac{z \ln z}{(1+z+z^2)^2}, \operatorname{res} f(e^{i2\pi/3}) = \lim_{z \rightarrow e^{i2\pi/3}} \frac{d}{dz} \frac{z \ln z}{(z - e^{i4\pi/3})^2} = -\frac{1}{3} - \frac{2}{9\sqrt{3}}\pi,$$

$$\operatorname{res} f(e^{i4\pi/3}) = \lim_{z \rightarrow e^{i4\pi/3}} \frac{d}{dz} \frac{z \ln z}{(z - e^{i2\pi/3})^2} = \frac{4}{9\sqrt{3}}\pi - \frac{1}{3}.$$

$$\text{原积分} = -\operatorname{res} f(e^{i2\pi/3}) - \operatorname{res} f(e^{i4\pi/3}) = \frac{2}{3} \left(1 - \frac{\sqrt{3}}{9}\pi \right).$$

$$(2) \text{ 令 } f(z) = \frac{\ln z}{z^3 + a^3},$$

$$\operatorname{res} f(ae^{i\pi/3}) = \lim_{z \rightarrow ae^{i\pi/3}} \frac{\ln z}{(z+a)(z - ae^{i5\pi/3})} = \frac{\sqrt{3}\pi - 3 \ln a - i(\pi + 3\sqrt{3} \ln a)}{18a^2},$$

$$\operatorname{res} f(-a) = \lim_{z \rightarrow ae^{i\pi}} \frac{\ln z}{z^2 - az + a^2} = \frac{\ln a + \pi i}{3a^2},$$

$$\operatorname{res} f(ae^{i5\pi/3}) = \lim_{z \rightarrow ae^{i5\pi/3}} \frac{\ln z}{(z+a)(z - ae^{i2\pi/3})} = \frac{-5\sqrt{3}\pi - 3 \ln a + i(3\sqrt{3} \ln a - 5\pi)}{18a^2}.$$

$$\text{原积分} = -\operatorname{res} f(ae^{i\pi/3}) - \operatorname{res} f(-a) - \operatorname{res} f(ae^{i5\pi/3}) = \frac{2\sqrt{3}\pi}{9a^2}.$$

$$(3) \text{ 令 } f(z) = \frac{\ln z}{(z+a)(z^2+b^2)}, \text{ 则 } \operatorname{res} f(-a) = \lim_{z \rightarrow ae^{i\pi}} \frac{\ln z}{z^2 + b^2} = \frac{\ln a + \pi i}{a^2 + b^2},$$

$$\operatorname{res} f(bi) = \lim_{z \rightarrow bae^{i\pi/2}} \frac{\ln z}{(z+a)(z+bi)} = \frac{a\pi - 2b \ln b - i(2a \ln b + b\pi)}{4b(a^2 + b^2)},$$

$$\operatorname{res} f(-bi) = \lim_{z \rightarrow bae^{3\pi/2}} \frac{\ln z}{(z+a)(z-bi)} = \frac{-3a\pi - 2b \ln b + i(2a \ln b - 3b\pi)}{4b(a^2 + b^2)}.$$

$$\text{原积分} = -\operatorname{res} f(-a) - \operatorname{res} f(-bi) - \operatorname{res} f(bi) = \frac{1}{a^2 + b^2} \left(\ln \frac{b}{a} + \frac{a\pi}{2b} \right).$$

$$(4) f(z) = \frac{\ln z}{(z^2 + a^2)(z^2 + b^2)}, \text{ 则 } \operatorname{res} f(ai) = \lim_{z \rightarrow ae^{i\pi/2}} \frac{\ln z}{(z+ai)(z^2 + b^2)} = \frac{-\pi + 2i \ln a}{4a(a^2 - b^2)},$$

$$\operatorname{res} f(-ai) = \lim_{z \rightarrow ae^{3\pi/2}} \frac{\ln z}{(z-ai)(z^2 + b^2)} = \frac{3\pi - 2i \ln a}{4a(a^2 - b^2)},$$

$$\operatorname{res} f(bi) = \lim_{z \rightarrow bae^{i\pi/2}} \frac{\ln z}{(z^2 + a^2)(z+bi)} = \frac{\pi - 2i \ln b}{4b(a^2 - b^2)},$$

$$\operatorname{res} f(-bi) = \lim_{z \rightarrow bae^{3\pi/2}} \frac{\ln z}{(z^2 + a^2)(z-bi)} = \frac{-3\pi + 2i \ln b}{4b(a^2 - b^2)}.$$

$$\text{原积分} = -\operatorname{res} f(ai) - \operatorname{res} f(-ai) - \operatorname{res} f(bi) - \operatorname{res} f(-bi) = \frac{\pi}{2ab(a+b)}.$$

43. 用类似于 143 题的方法证明 $\int_0^{\infty} f(x) \ln x dx = -\frac{1}{2} \operatorname{Re} \sum_{\text{全平面}} \operatorname{res} \{f(z)(\ln z)^2\}$.

$$\int_0^{\infty} f(x) dx = -\frac{1}{2\pi} \operatorname{Im} \sum_{\text{全平面}} \operatorname{res} \{f(z)(\ln z)^2\}, \quad 0 \leq \arg z \leq 2\pi. \quad \text{其中 } f(x) \text{ 满足和第 143}$$

题中 $\frac{P(z)}{Q(z)}$ 同样的要求。

证: 令 $F(z) = f(z)(\ln z)^2$, 取与 143 题相同的积分路径, 规定 $0 \leq \arg z < 2\pi$, 同样有

$$\lim_{\delta \rightarrow 0} \int_{C_{\delta}} F(z) dz = 0, \quad \lim_{R \rightarrow \infty} \int_{C_R} F(z) dz = 0. \quad \text{围线积分为}$$

$$\int_{\delta}^R f(x)(\ln x)^2 dx + \int_R^{\delta} f(x)(\ln x + 2\pi i)^2 dx + \left(\int_{C_{\delta}} + \int_{C_R} \right) F(z) dz = 2\pi i \sum_{\text{全平面}} \operatorname{res} \{F(z)\},$$

$$\text{即 } 4\pi^2 \int_{\delta}^R f(x) dx - 4\pi i \int_{\delta}^R f(x) \ln x dx + \left(\int_{C_{\delta}} + \int_{C_R} \right) F(z) dz = 2\pi i \sum_{\text{全平面}} \operatorname{res} \{F(z)\}.$$

$$\text{令上式 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } \pi \int_0^{\infty} f(x) dx - i \int_0^{\infty} f(x) \ln x dx = \frac{1}{2} i \sum_{\text{全平面}} \operatorname{res} \{f(z)(\ln z)^2\},$$

比较两边实部和虚部即得证。

44. 利用上题结论计算下列积分: (1) $\int_0^{\infty} \frac{\ln x}{x^2 + a^2} dx, \quad a > 0$; (2) $\int_0^{\infty} \frac{\ln x}{(x+a)(x+b)} dx,$

$$b > a > 0; \quad (3) \int_0^{\infty} \frac{\ln x}{(x+a)^2} dx, \quad a > 0; \quad (4) \int_0^{\infty} \frac{\ln x}{(x+a)^2 + b^2} dx, \quad a, b \text{ 均为正数}.$$

$$(1) \text{ 令 } f(z) = \frac{(\ln z)^2}{z^2 + a^2}, \text{ 则 } \operatorname{res} f(ai) = \lim_{z \rightarrow ae^{i\pi/2}} \frac{(\ln z)^2}{z + ai} = \frac{\pi \ln a}{2a} + i \frac{\pi^2 - 4 \ln^2 a}{8a},$$

$$\operatorname{res} f(-ai) = \lim_{z \rightarrow ae^{3\pi/2}} \frac{(\ln z)^2}{z - ai} = -\frac{3\pi \ln a}{2a} + i \frac{4 \ln^2 a - 9\pi^2}{8a}.$$

$$\text{原积分} = -\frac{1}{2} \{ \operatorname{Re} [\operatorname{res} f(ai)] + \operatorname{Re} [\operatorname{res} f(-ai)] \} = \frac{\pi \ln a}{2a}.$$

$$(2) \text{ 令 } f(z) = \frac{(\ln z)^2}{(z+a)(z+b)}, \text{ 则 } f(-a) = \lim_{z \rightarrow ae^{i\pi}} \frac{(\ln z)^2}{z+b} = \frac{\ln^2 a - \pi^2}{b-a} + i \frac{2\pi \ln a}{b-a},$$

$$f(-b) = \lim_{z \rightarrow be^{i\pi}} \frac{(\ln z)^2}{z+a} = \frac{\pi^2 - \ln^2 b}{b-a} - i \frac{2\pi \ln b}{b-a}.$$

$$\text{原积分} = -\frac{1}{2} \{ \operatorname{Re} [\operatorname{res} f(-a)] + \operatorname{Re} [\operatorname{res} f(-b)] \} = \frac{\ln^2 b - \ln^2 a}{2(b-a)} = \frac{\ln a \ln b}{2(b-a)}.$$

$$(3) \text{ 令 } f(z) = \frac{(\ln z)^2}{(z+a)^2}, \text{ 则 } \operatorname{res} f(-a) = \lim_{z \rightarrow ae^{i\pi}} \frac{d}{dz} (\ln z)^2 = -\frac{2 \ln a}{a} - i \frac{2\pi}{a}.$$

$$\text{原积分} = -\frac{1}{2} \operatorname{Re} [\operatorname{res} f(-a)] = \frac{\ln a}{a}.$$

$$(4) \text{ 令 } f(z) = \frac{(\ln z)^2}{(z+a)^2 + b^2}, \text{ 则 } \operatorname{res} f(-a+bi) = \lim_{z \rightarrow \sqrt{a^2+b^2}e^{i(\pi-\tan^{-1}\frac{b}{a})}} \frac{(\ln z)^2}{z+a+bi}$$

$$= \frac{1}{2b} \left(\pi - \tan^{-1} \frac{b}{a} \right) \ln(a^2 + b^2) + i \frac{1}{2b} \left[\left(\pi - \tan^{-1} \frac{b}{a} \right)^2 - \frac{1}{4} \ln^2(a^2 + b^2) \right],$$

$$\operatorname{res} f(-a-bi) = \lim_{z \rightarrow \sqrt{a^2+b^2}e^{i(\pi+\tan^{-1}\frac{b}{a})}} \frac{(\ln z)^2}{z+a-bi}$$

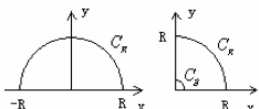
$$= -\frac{1}{2b} \left(\pi + \tan^{-1} \frac{b}{a} \right) \ln(a^2 + b^2) + i \frac{1}{2b} \left[\frac{1}{4} \ln^2(a^2 + b^2) - \left(\pi + \tan^{-1} \frac{b}{a} \right)^2 \right].$$

$$\text{原积分} = -\frac{1}{2} \{ \operatorname{Re} [\operatorname{res} f(-a+bi)] + \operatorname{Re} [\operatorname{res} f(-a-bi)] \} = \frac{1}{2b} \tan^{-1} \frac{b}{a} \ln(a^2 + b^2).$$

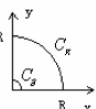
45 按照指定的积分围道, 考虑适当的复变积分, 计算下列定积分: **(185)**

$$(1) \int_0^{\infty} \frac{(1+x^2) \cos ax}{1+x^2+x^4} dx, \int_0^{\infty} \frac{x \sin ax}{1+x^2+x^4} dx, \quad a > 0; \quad (2) \int_0^{\infty} \frac{\cos x - e^x}{x} dx; \quad (3) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx,$$

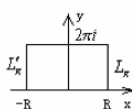
$$0 < a < 1; \quad (4) \int_0^1 \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3} dx.$$



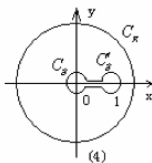
(1)



(2)



(3)



(4)

$$(1) \text{ 令 } f(z) = \frac{(1+z^2)e^{iaz}}{1+z^2+z^4}, \text{ 则}$$

$$\operatorname{res} f(e^{i\pi/3}) = \lim_{z \rightarrow e^{i\pi/3}} \frac{(1+z^2)e^{iaz}}{(z-e^{-i\pi/3})(z^2+z+1)} = -\frac{1}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \left(-\sin \frac{a}{2} + i \cos \frac{a}{2} \right),$$

$$\operatorname{res} f(e^{2\pi/3}) = \lim_{z \rightarrow e^{2i\pi/3}} \frac{(1+z^2)e^{iaz}}{(z^2-z+1)(z-e^{-i2\pi/3})} = -\frac{1}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \left(\sin \frac{a}{2} + i \cos \frac{a}{2} \right),$$

对围道积分取 $R \rightarrow \infty$ 得

$$\int_0^{\infty} \frac{(1+x^2) \cos ax}{1+x^2+x^4} dx = \frac{1}{2} \operatorname{Re} \{ 2\pi i [\operatorname{res} f(e^{i\pi/3}) + \operatorname{res} f(e^{2\pi/3})] \}$$

$$= -\pi \operatorname{Im} [\operatorname{res} f(e^{i\pi/3}) + \operatorname{res} f(e^{2\pi/3})] = \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \cos \frac{a}{2}.$$

$$\text{令 } f(z) = \frac{ze^{iaz}}{1+z^2+z^4}, \text{ 则}$$

$$\operatorname{res} f(e^{i\pi/3}) = \lim_{z \rightarrow e^{i\pi/3}} \frac{ze^{iaz}}{(z-e^{-i\pi/3})(z^2+z+1)} = \frac{1}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \left(\sin \frac{a}{2} - i \cos \frac{a}{2} \right),$$

$$\operatorname{res} f(e^{2\pi/3}) = \lim_{z \rightarrow e^{2i\pi/3}} \frac{ze^{iaz}}{(z^2-z+1)(z-e^{-i2\pi/3})} = \frac{1}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \left(\sin \frac{a}{2} + i \cos \frac{a}{2} \right)$$

对围道积分取 $R \rightarrow \infty$ 得

$$\int_0^{\infty} \frac{x \sin ax}{1+x^2+x^4} dx = \frac{1}{2} \operatorname{Im} \{ 2\pi i [\operatorname{res} f(e^{i\pi/3}) + \operatorname{res} f(e^{2\pi/3})] \}$$

$$= \pi \operatorname{Re} [\operatorname{res} f(e^{i\pi/3}) + \operatorname{res} f(e^{2\pi/3})] = \frac{\pi}{\sqrt{3}} e^{-\frac{\sqrt{3}}{2}a} \sin \frac{a}{2}.$$

$$(2) \text{ 令 } f(z) = \frac{e^{iz}}{z}, \text{ 则 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0 \quad (\text{由 Jordan 引理的证明过程可看出对于上半}$$

平面张角小于 π 的弧该定理仍成立), $\lim_{\delta \rightarrow 0} \int_{C_{\delta}} f(z) dz = -\frac{\pi i}{2} \lim_{z \rightarrow 0} e^{iz} = -\frac{\pi i}{2}$, 围道积分为:

$$\int_{\delta}^R \frac{e^{ix}}{x} dx + \int_R^{\delta} \frac{e^{-iy}}{y} dy + \left(\int_{C_R} + \int_{C_{\delta}} \right) f(z) dz = 0, \quad \text{令 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } \int_0^{\infty} \frac{e^{ix} - e^{-x}}{x} dx = \frac{\pi i}{2},$$

$$\text{两边取实部即可得 } \int_0^{\infty} \frac{\cos x - e^{-x}}{x} dx = 0.$$

$$(3) \text{ 令 } f(z) = \frac{e^{az}}{1+e^z}, \text{ 则 } \operatorname{res} f(\pi i) = \lim_{z \rightarrow \pi i} \frac{e^{az}}{(1+e^z)'} = -e^{ia\pi}. \quad \text{类似于 141 题第 (6)}$$

小题作法可证 $\lim_{R \rightarrow \infty} \int_{L_R} f(z) dz = 0$ 和 $\lim_{R \rightarrow \infty} \int_{L_{\delta}} f(z) dz = 0$ (也可证 $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$ 收敛).

$$\text{围道积分为 } \int_{-R}^R \frac{e^{ax}}{1+e^x} dx + e^{2ia\pi} \int_R^{-R} \frac{e^{ax}}{1+e^x} dx + \left(\int_{L_R} + \int_{L_{\delta}} \right) f(z) dz = -2\pi i e^{ia\pi}.$$

$$\text{令 } R \rightarrow \infty \text{ 可得 } \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{-2\pi i e^{ia\pi}}{1-e^{2ia\pi}} = \frac{\pi}{\sin a\pi}.$$

$$(4) \text{ 令 } f(z) = \frac{\sqrt[4]{z(1-z)^3}}{(1+z)^3}, \text{ 规定割线上岸 } \arg z = 0, \arg(1-z) = 0, \text{ 则割线上岸的积分}$$

$$\text{为 } \int_{\delta}^{1-\delta} \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3} dx, \quad \text{割线下岸积分为 } \int_{1-\delta}^{\delta} \frac{\sqrt[4]{x(1-x)^3 e^{-2i\pi}}}{(1+x)^3} dx = -i \int_{1-\delta}^{1-\delta} \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3} dx.$$

$$\text{由于 } \lim_{z \rightarrow \infty} z f(z) = \lim_{z \rightarrow \infty} \frac{z \sqrt[4]{z(1-z)^3}}{(1+z)^3} = 0 \text{ 与幅角无关, 所以 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

$$\lim_{\delta \rightarrow 0} \int_{C_{\delta}} f(z) dz = -2\pi i \lim_{z \rightarrow 0} \frac{z \sqrt[4]{z(1-z)^3}}{(1+z)^3} = 0 \quad (\text{与幅角无关}),$$

$$\lim_{\delta \rightarrow 0} \int_{C_{\delta}} f(z) dz = -2\pi i \lim_{z \rightarrow 1} \frac{(z-1) \sqrt[4]{z(1-z)^3}}{(1+z)^3} = 0 \quad (\text{与幅角无关}),$$

$$\operatorname{res} f(-1) = \frac{1}{2} \lim_{z \rightarrow -e^{i\pi}} \frac{d^2}{dz^2} [z^{1/4} (1-z)^{3/4}]$$

$$= -\frac{3}{32} \lim_{z \rightarrow -e^{i\pi}} [z^{-7/4} (1-z)^{3/4} + 2z^{-3/4} (1-z)^{-1/4} + z^{1/4} (1-z)^{-5/4}] = -\frac{3}{64} 2^{-1/4} e^{i\pi/4}.$$

围道积分为

$$(1-i) \int_{\delta}^{1-\delta} \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3} dx + \left(\int_{C_2} + \int_{C_{\delta}} + \int_{C_{\delta}} \right) f(z) dz = 2\pi i \operatorname{res} f(-1) = -i \frac{3}{64} 2^{3/4} \pi e^{i\pi/4},$$

$$\text{令 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } \int_0^1 \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3} dx = -i \frac{3}{64} 2^{3/4} \pi \frac{e^{i\pi/4}}{1-i} = \frac{3}{64} 2^{1/4} \pi.$$

按照指定的被积函数, 选择适当的积分围道, 计算下列积分:

- (1) $\int_0^\pi \frac{\cos n\varphi}{a - ib \cos \varphi} d\varphi$, $a > 0, b > 0$, 被积函数为 $\frac{z^n}{bz^2 + 2iaz + b}$;
- (2) $\int_0^\infty \frac{x^b}{1+x^2} \cos\left(ax - \frac{1}{2}bx\right) dx$, $a \geq 0, -1 < b < 1$, 被积函数为 $\frac{e^{iaz} z^b}{1+z^2}$;
- (3) $\int_0^\infty \frac{dx}{x[(\ln x)^2 + \pi^2]}$, 被积函数为 $\frac{1}{z \ln z}$;
- (4) $\int_0^\infty \frac{x \tan^{-1} x}{(1+2x^2)^2} dx$, 被积函数为 $\frac{z \ln(1-iz)}{(1+2z^2)^2}$;
- (5) $\int_0^\infty \frac{\cos(\ln x)}{1+x^2} dx$, 被积函数为 $\frac{z^i}{z^2-1}$;
- (6) $\int_0^{\pi/2} \frac{r \sin 2\theta}{1-2r \cos 2\theta + r^2} d\theta$, 被积函数为 $\frac{2zr}{z^2(1+r)^2 + (1-r)^2} \frac{\ln(1-iz)}{1+z^2}$.

$$\begin{aligned} (1) \int_{|z|=R} \frac{z^n}{bz^2 + 2iaz + b} dz &= i \int_0^{2\pi} \frac{e^{in\varphi}}{be^{2i\varphi} + 2iae^{i\varphi} + b} e^{i\varphi} d\varphi = i \int_0^{2\pi} \frac{e^{in\varphi} d\varphi}{be^{i\varphi} + 2ia + be^{-i\varphi}} \\ &= \frac{1}{2} \int_0^{2\pi} \frac{e^{in\varphi} d\varphi}{a - ib \cos \varphi} = \frac{1}{2} \int_0^\pi \frac{e^{in\varphi} d\varphi}{a - ib \cos \varphi} + \frac{1}{2} \int_\pi^{2\pi} \frac{e^{in\varphi} d\varphi}{a - ib \cos \varphi} \\ &= \frac{1}{2} \int_0^\pi \frac{e^{in\varphi} d\varphi}{a - ib \cos \varphi} + \frac{1}{2} \int_0^\pi \frac{e^{-in\theta} d\theta}{a - ib \cos \theta} \quad (\text{作代换 } \theta = 2\pi - \varphi) \\ &= \int_0^\pi \frac{\cos n\varphi}{a - ib \cos \varphi} d\varphi \end{aligned}$$

$$\text{记 } f(z) = \frac{z^n}{bz^2 + 2iaz + b} = \frac{z^n}{b(z-z_1)(z-z_2)}, \text{ 其中 } z_1 = \left(\sqrt{\frac{a^2}{b^2} + 1} - \frac{a}{b} \right) i,$$

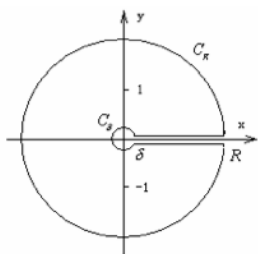
$$z_2 = -\left(\sqrt{\frac{a^2}{b^2} + 1} + \frac{a}{b} \right) i, \text{ 显然 } z_1 \text{ 在单位圆内, } z_2 \text{ 在单位圆外.}$$

$$\text{res } f(z_1) = \lim_{z \rightarrow z_1} \frac{z^n}{b(z-z_2)} = \frac{i^n \left(\sqrt{\frac{a^2}{b^2} + 1} - \frac{a}{b} \right)^n}{2bi \sqrt{\frac{a^2}{b^2} + 1}} = \frac{1}{2i \sqrt{a^2 + b^2}} \left(\frac{i}{b} \right)^n \left(\sqrt{a^2 + b^2} - a \right)^n,$$

$$\text{原积分} = 2\pi i \text{res } f(z_1) = \frac{\pi}{\sqrt{a^2 + b^2}} \left(\frac{i}{b} \right)^n \left(\sqrt{a^2 + b^2} - a \right)^n.$$

$$(2) \text{ 令 } f(z) = \frac{e^{iaz} z^b}{1+z^2}, \text{ 规定 } 0 \leq \arg z \leq 2\pi, \text{ 则 } \text{res } f(i) = \lim_{z \rightarrow e^{i\pi/2}} \frac{e^{iaz} z^b}{z+i} = \frac{e^{-a}}{2i} e^{i\frac{b}{2}\pi},$$

$$\text{res } f(-i) = \lim_{z \rightarrow -e^{i\pi/2}} \frac{e^{iaz} z^b}{z-i} = -\frac{e^a}{2i} e^{i\frac{3b}{2}\pi}. \text{ 取如下积分路径:}$$



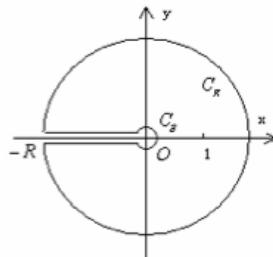
$$\begin{aligned} \text{易得 } \int_0^\infty \frac{x^b e^{iax}}{1+x^2} dx &= \frac{2\pi i}{(1-e^{2i\pi})} [\text{res } f(i) + \text{res } f(-i)] = -i\pi \frac{\text{sh}\left(a + i\frac{b}{2}\pi\right)}{\sin b\pi} \\ &= \pi \left(\frac{\text{ch } a}{2 \cos \frac{b\pi}{2}} - i \frac{\text{sh } a}{2 \sin \frac{b\pi}{2}} \right). \end{aligned}$$

$$\text{所以 } \int_0^\infty \frac{x^b \cos ax}{1+x^2} dx = \pi \frac{\text{ch } a}{2 \cos \frac{b\pi}{2}}, \quad \int_0^\infty \frac{x^b \sin ax}{1+x^2} dx = -\pi \frac{\text{sh } a}{2 \sin \frac{b\pi}{2}}.$$

$$\text{原积分} = \cos \frac{b\pi}{2} \int_0^\infty \frac{x^b \cos ax}{1+x^2} dx + \sin \frac{b\pi}{2} \int_0^\infty \frac{x^b \sin ax}{1+x^2} dx = \frac{\pi}{2} (\text{ch } a - \text{sh } a) = \frac{\pi}{2} e^{-a}.$$

$$(3) \text{ 令 } f(z) = \frac{1}{z \ln z}, \text{ 规定 } -\pi \leq \arg z \leq \pi, \text{ res } f(1) = \lim_{z \rightarrow e^{i0}} \frac{1}{z(\ln z)'} = 1. \text{ 积分路径如}$$

下:



$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 2\pi i \lim_{z \rightarrow \infty} z f(z) = 2\pi i \lim_{z \rightarrow \infty} \frac{1}{\ln z} = 0,$$

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = -2\pi i \lim_{z \rightarrow 0} z f(z) = -2\pi i \lim_{z \rightarrow 0} \frac{1}{\ln z} = 0,$$

$$\text{割线上岸有 } z = re^{i\pi}, \text{ 积分为 } \int_R^\delta \frac{1}{re^{i\pi} (\ln r + \pi i)} d(re^{i\pi}) = -\int_\delta^R \frac{1}{x(\ln x + \pi i)} dx, \text{ 同样可得}$$

$$\text{线下岸积分为 } \int_\delta^R \frac{1}{x(\ln x - \pi i)} dx. \text{ 围道积分为:}$$

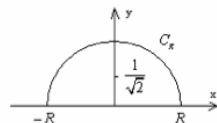
$$\int_\delta^R \frac{1}{x} \left(\frac{1}{\ln x - \pi i} - \frac{1}{\ln x + \pi i} \right) dx + \left(\int_{C_R} + \int_{C_\delta} \right) f(z) dz = 2\pi i.$$

$$\text{令 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } 2\pi i \int_0^\infty \frac{dx}{x[(\ln x)^2 + \pi^2]} = 2\pi i, \text{ 即 } \int_0^\infty \frac{dx}{x[(\ln x)^2 + \pi^2]} = 1.$$

$$(4) \text{ 令 } f(z) = \frac{z \ln(1-iz)}{4\left(z^2 + \frac{1}{2}\right)^2} = \frac{z \ln(1-iz)}{4\left(z + \frac{1}{\sqrt{2}}i\right)^2 \left(z - \frac{1}{\sqrt{2}}i\right)^2},$$

$$\begin{aligned} \text{res } f\left(\frac{1}{\sqrt{2}}i\right) &= \lim_{z \rightarrow \frac{1}{\sqrt{2}}i} \frac{d}{dz} \frac{z \ln(1-iz)}{4\left(z + \frac{1}{\sqrt{2}}i\right)^2} = \lim_{z \rightarrow \frac{1}{\sqrt{2}}i} \frac{\left[\ln(1-iz) - \frac{iz}{1-iz} \right] \left(z + \frac{1}{\sqrt{2}}i \right) - 2z \ln(1-iz)}{4\left(z + \frac{1}{\sqrt{2}}i\right)^3} \\ &= -\frac{1}{8}(\sqrt{2}-1). \end{aligned}$$

$-i$ 和 ∞ 是 $f(z)$ 的枝点, 可沿虚轴从 $-i$ 向下延伸到 ∞ 作为割线, 取如下积分路径 (在单值分枝内):



$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = \pi i \lim_{z \rightarrow \infty} \frac{z^2 \ln(1-iz)}{(1+2z^2)^2} = 0. \text{ 围道积分为:}$$

$$\int_{-R}^R \frac{x \ln(1-ix)}{(1+2x^2)^2} dx + \int_0^R \frac{x \ln(1-ix)}{(1+2x^2)^2} dx + \int_{C_R} f(z) dz = 2\pi i \text{res } f\left(\frac{1}{\sqrt{2}}i\right) = -i\frac{\pi}{4}(\sqrt{2}-1).$$

$$\text{由于 } \int_{-R}^0 \frac{x \ln(1-ix)}{(1+2x^2)^2} dx = \int_R^0 \frac{y \ln(1+iy)}{(1+2y^2)^2} dy = -\int_0^R \frac{x \ln(1+ix)}{(1+2x^2)^2} dx,$$

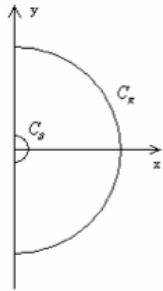
$$\text{所以 } \int_{-R}^0 \frac{x \ln(1-ix)}{(1+2x^2)^2} dx + \int_0^R \frac{x \ln(1-ix)}{(1+2x^2)^2} dx = \int_0^R \frac{x \ln\left(\frac{1-ix}{1+ix}\right)}{(1+2x^2)^2} dx = -2i \int_0^R \frac{x \tan^{-1} x}{(1+2x^2)^2} dx.$$

$$\text{令围道积分 } R \rightarrow \infty \text{ 得 } -2i \int_0^\infty \frac{x \tan^{-1} x}{(1+2x^2)^2} dx = -i\frac{\pi}{4}(\sqrt{2}-1),$$

$$\text{即 } \int_0^\infty \frac{x \tan^{-1} x}{(1+2x^2)^2} dx = \frac{\pi}{8}(\sqrt{2}-1).$$

(5) 令 $f(z) = \frac{z^i}{z^2-1} = \frac{e^{i \ln z}}{z^2-1}$, 0 和 ∞ 是其枝点, 以负实轴作为割线, 规定

$-\pi < \arg z \leq \pi$, 则 $\operatorname{res} f(1) = \lim_{z \rightarrow e^{i0}} \frac{e^{i \ln z}}{z+1} = \frac{1}{2}$ 。可取如下积分路径:



正虚轴上有 $z = re^{i\pi/2}$, 积分为 $\int_R^\delta \frac{e^{i(\ln r + \frac{\pi}{2}i)}}{-r^2-1} d(re^{i\pi/2}) = ie^{-\pi/2} \int_\delta^R \frac{e^{i \ln x}}{x^2+1} dx$,

负虚轴上 $z = re^{-i\pi/2}$ 积分为 $ie^{\pi/2} \int_\delta^R \frac{e^{i \ln x}}{x^2+1} dx$ 。围道积分为

$$2i \operatorname{ch} \frac{\pi}{2} \int_\delta^R \frac{e^{i \ln x}}{x^2+1} dx + \left(\int_{C_R} + \int_{C_\rho} \right) f(z) dz = 2\pi i \operatorname{res} f(1) = \pi i。$$

由于 $zf(z) = \frac{e^{(1+i) \ln z}}{z^2-1} = \frac{e^{(1+i)(\ln|z|+i \arg z)}}{z^2-1} = \frac{e^{i(\ln|z|-\arg z)}}{z^2-1} e^{i(\ln|z|+\arg z)}$, $|z| \rightarrow 0$ 时 $\ln|z| \rightarrow -\infty$,

$e^{i(\ln|z|-\arg z)} \rightarrow 0$ (与 $\arg z$ 无关), 所以 $\lim_{\delta \rightarrow 0} \int_{C_\rho} f(z) dz = 0$ 。

$$\lim_{z \rightarrow \infty} zf(z) = \lim_{z \rightarrow \infty} \frac{z^{1+i}}{z^2-1} = \lim_{z \rightarrow \infty} \frac{(1+i)z^i}{2z} = \frac{1+i}{2} \lim_{z \rightarrow \infty} z^{-1+i} = \frac{1+i}{2} \lim_{z \rightarrow \infty} e^{(-1+i)(\ln|z|+i \arg z)}$$

$$= \frac{1+i}{2} \lim_{z \rightarrow \infty} e^{(-\ln|z|-\arg z)} e^{i(\ln|z|-\arg z)} = 0 \text{ (与 } \arg z \text{ 无关),}$$

$$\text{所以 } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0。 \text{令围道积分 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } \int_0^\pi \frac{e^{i \ln x}}{x^2+1} dx = \frac{\pi}{2 \operatorname{ch} \frac{\pi}{2}}。$$

八. 解析延拓

47 证明 $f_1(z) = 1 + az + a^2 z^2 + \dots$ 与 $f_2(z) = \frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2 z^2}{(1-z)^3} - \dots$ 互为解

析延拓。

当 $|az| < 1$, 即 $|z| < \frac{1}{|a|}$ 时, $f_1(z) = \frac{1}{1-az}$ 。

$$\text{当 } \left| \frac{(1-a)z}{1-z} \right| < 1 \text{ 时, } f_2(z) = \frac{1}{1-z} \frac{1}{1 + \frac{(1-a)z}{1-z}} = \frac{1}{1-az}。$$

将 $\left| \frac{(1-a)z}{1-z} \right| < 1$ 化成 $(|1-a|^2 - 1)z\bar{z} + z + \bar{z} - 1 < 0$, 这是圆内(外)方程(见习题 02 第 47

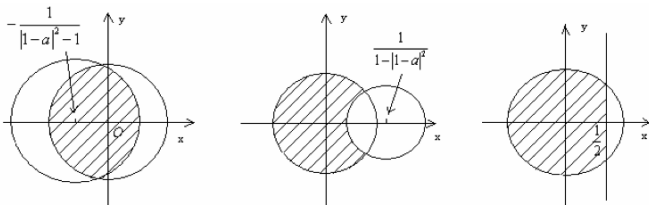
题), 圆心为 $c = -\frac{1}{|1-a|^2 - 1}$, 半径为 $R = \frac{|1-a|}{||1-a|^2 - 1|}$ 。

若 $|1-a| > 1$, 这是圆内方程, 圆心在负实轴, 且 $R > |c|$; 若 $|1-a| < 1$, 这是圆外方程, 圆心在正实轴, 且 $R < |c|$ 。

当 $|1-a| = 1$, $(|1-a|^2 - 1)z\bar{z} + z + \bar{z} - 1 < 0$ 化为 $x < \frac{1}{2}$ 。

这三种情况 $(|1-a|^2 - 1)z\bar{z} + z + \bar{z} - 1 < 0$ 都与 $|z| < \frac{1}{|a|}$ 有交集, 如下图所示。

在交集上有 $f_1(z) = f_2(z)$, 所以两者互为解析延拓。



48 证明级数 $\sum_{n=1}^\infty \left(\frac{1}{1-z^{n+1}} - \frac{1}{1-z^n} \right)$ 在区域 $|z| < 1$ 与 $|z| > 1$ 内分别代表两个解析函数, 但不

互为解析延拓。

$$S_N(z) = \sum_{n=1}^N \left(\frac{1}{1-z^{n+1}} - \frac{1}{1-z^n} \right) = -\frac{1}{1-z} + \frac{1}{1-z^2} - \frac{1}{1-z^2} + \frac{1}{1-z^3} - \dots - \frac{1}{1-z^N} + \frac{1}{1-z^{N+1}} \\ = \frac{1}{1-z^{N+1}} - \frac{1}{1-z}。$$

$|z| < 1$ 时, $S(z) = \lim_{N \rightarrow \infty} S_N(z) = 1 - \frac{1}{1-z} = -\frac{z}{1-z}$, $|z| > 1$ 时, $S(z) = -\frac{1}{1-z}$ 。显然两者不互为解析延拓。

49 已知: $f(z) = \sum_{n=0}^\infty z^{2^n} = z + z^2 + z^4 + z^8 + \dots$, $|z| < 1$ 。(1) 证明: $z=1$ 是 $f(z)$ 的奇

点;(2) 证明: $f(z) = z + f(z^2)$, 因此 $z^2=1$ 的根也都是 $f(z)$ 的奇点;(3) 类似的证明:

$z^{2^k}=1$ 的 2^k 个根也是 $f(z)$ 的奇点, k 为任意正整数;(4) 由此证明: 不可能将 $f(z)$ 解析延拓到单位圆外。

(1) $f(1) = \sum_{n=0}^\infty 1 \rightarrow \infty$, 即 $z=1$ 是 $f(z)$ 的奇点。

$$(2) f(z^2) = \sum_{n=0}^\infty z^{2^{n+1}} = \sum_{n=1}^\infty z^{2^n} = f(z) - z。$$

$$(3) f(z) = z + f(z^2) = z + z^2 + f(z^4) \\ = \dots = z + z^2 + z^4 + \dots + z^{2^{k-1}} + f(z^{2^k})$$

(4) 对于任意包含单位圆上一段弧的区域, 总存在 N , 使得 $z^{2^N}=1$ 的某个根落入该区域, 所以不可能将 $f(z)$ 解析延拓到单位圆外。

九. Gamma 函数

50 将下列连乘积用 Γ 函数表示出来: (1) $(2n)!!$; (2) $(2n-1)!!$;

$$(3) (1+\rho)(2+\rho)\dots(n+\rho); (4) [n(n+1)-\rho(\rho+1)][(n-1)n-\rho(\rho+1)]\dots[0-\rho(\rho+1)]。$$

$$(1) (2n)!! = (2n)(2n-2)(2n-4)\dots 2 = 2^n n(n-1)(n-2)\dots 1 = 2^n n! = 2^n \Gamma(n+1)。$$

$$(2) (2n-1)!! = (2n-1)(2n-3)\dots 1 = \frac{(2n)(2n-1)(2n-2)(2n-3)\dots 2 \cdot 1}{(2n)!!}$$

$$= \frac{(2n)!}{(2n)!!} = \frac{\Gamma(2n+1)}{2^n \Gamma(n+1)}。$$

$$(3) \Gamma(\rho+n+1) = (\rho+n)\Gamma(\rho+n) = (\rho+n)(\rho+n-1)\Gamma(\rho+n-1) = \dots \\ = (\rho+n)(\rho+n-1)\dots(\rho+1)\Gamma(\rho+1)$$

$$\text{所以 } (1+\rho)(2+\rho)\dots(n+\rho) = \frac{\Gamma(\rho+n+1)}{\Gamma(\rho+1)}。$$

$$(4) n(n+1)-\rho(\rho+1) = -[\rho^2 + \rho - n(n+1)] = (\rho+n+1)(n-\rho),$$

$$\text{原式} = (\rho+n+1)(\rho+n)\dots(\rho+1)\cdot(n-\rho)(n-1-\rho)\dots(-\rho) = \frac{\Gamma(\rho+n+2)}{\Gamma(\rho+1)} \frac{\Gamma(n+1-\rho)}{\Gamma(-\rho)}。$$

51 设 $\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, 证明: (1) $\psi(z+1) = \frac{1}{z} + \psi(z)$;

$$(2) \psi(z+n) - \psi(z) = \frac{1}{z} + \frac{1}{z+1} + \dots + \frac{1}{z+n-1}; (3) \psi(1-z) - \psi(z) = \pi \cot(\pi z);$$

$$(4) 2\psi(2z) - \psi(z) - \psi\left(z + \frac{1}{2}\right) = 2 \ln 2。$$

(1) 对 $\Gamma(z+1) = z\Gamma(z)$ 两边取对数, 再微商即可证。

(2) 由 $\Gamma(z+n) = (z+n-1)(z+n-2)\dots(z+1)z\Gamma(z)$ 可证。

$$(3) \text{由 } \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \text{ 可得。}$$

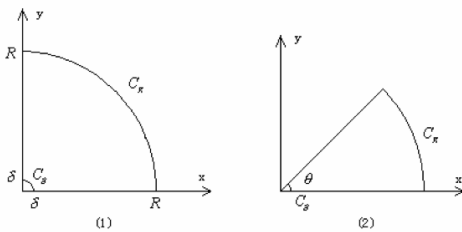
$$(4) \text{由 } \Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \text{ 可得。}$$

52 计算下列积分：(1) $\int_0^{\infty} x^{-\alpha} \sin x dx$, $0 < \alpha < 2$, $\int_0^{\infty} x^{-\alpha} \cos x dx$, $0 < \alpha < 1$;

$$(2) \int_0^{\infty} x^{\alpha-1} e^{-x \cos \theta} \cos(x \sin \theta) dx, \int_0^{\infty} x^{\alpha-1} e^{-x \cos \theta} \sin(x \sin \theta) dx, \quad \alpha > 0, |\theta| < \frac{\pi}{2}.$$

(1) 令 $f(z) = z^{-\alpha} e^{-z}$, 规定 $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, 在下面左图路径上积分:

19.4)



$$\int_{\delta}^R r^{-\alpha} e^{-r} dr + \int_R^{\delta} (re^{i\pi/2})^{-\alpha} e^{-ir} d(re^{i\pi/2}) + \left(\int_{C_R} + \int_{C_{\delta}} \right) f(z) dz = 0$$

当 $0 < \alpha < 1$ 时, $0 < 1 - \alpha < 1$, $\lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} z^{1-\alpha} e^{-z} = 0$, 所以 $\lim_{\delta \rightarrow 0} \int_{C_{\delta}} f(z) dz = 0$.

又因为 $\operatorname{Re} z > 0$, 所以 $\lim_{z \rightarrow \infty} z f(z) = \lim_{z \rightarrow \infty} z^{1-\alpha} e^{-z} = 0$, 因此 $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

$$\text{令围道积分 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } \int_0^{\infty} x^{-\alpha} e^{-ix} dx = -ie^{\frac{i\alpha\pi}{2}} \int_0^{\infty} x^{-\alpha} e^{-x} dx = -ie^{\frac{i\alpha\pi}{2}} \Gamma(1-\alpha).$$

$$\text{取实部和虚部既可得 } 0 < \alpha < 1 \text{ 时有 } \int_0^{\infty} x^{-\alpha} \sin x dx = \Gamma(1-\alpha) \cos \frac{\alpha\pi}{2},$$

$$\int_0^{\infty} x^{-\alpha} \cos x dx = \Gamma(1-\alpha) \sin \frac{\alpha\pi}{2}.$$

157 题第 (2) 小题已证明 $\int_0^{\infty} x^{-\alpha} \sin x dx$ 在区间 $\alpha \in [\delta, 2-\delta]$ ($\forall \delta > 0$) 上是一致收敛的,

所以它在 $\alpha \in (\delta, 2-\delta)$ 上是 α 的解析函数, 由 δ 的任意性, 它是 $(0, 2)$ 上的解析函数. 由

$$\text{解析延拓原理, } 0 < \alpha < 2 \text{ 时有 } \int_0^{\infty} x^{-\alpha} \sin x dx = \Gamma(1-\alpha) \cos \frac{\alpha\pi}{2}.$$

(2) 取 $f(z) = z^{\alpha-1} e^{-z}$ 在上面右图路径上的积分,

$$\int_{\delta}^R r^{\alpha-1} e^{-r} dr + \int_R^{\delta} (re^{i\theta})^{\alpha-1} e^{-r(\cos \theta + i \sin \theta)} d(re^{i\theta}) + \left(\int_{C_R} + \int_{C_{\delta}} \right) f(z) dz = 0$$

由于 $\alpha > 0$, $\lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} z^{\alpha} e^{-z} = 0$, 所以 $\lim_{\delta \rightarrow 0} \int_{C_{\delta}} f(z) dz = 0$.

因为 $\operatorname{Re} z > 0$ ($|\theta| < \frac{\pi}{2}$), 所以 $\lim_{z \rightarrow \infty} z f(z) = \lim_{z \rightarrow \infty} z^{\alpha} e^{-z} = 0$, 因此 $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

$$\text{令围道积分 } \delta \rightarrow 0, R \rightarrow \infty \text{ 得 } \int_0^{\infty} x^{\alpha-1} e^{-x \cos \theta} e^{-ix \sin \theta} dx = e^{-i\alpha\theta} \Gamma(\alpha)$$

$$\text{取实部和虚部得 } \int_0^{\infty} x^{\alpha-1} e^{-x \cos \theta} \cos(x \sin \theta) dx = \Gamma(\alpha) \cos \alpha \theta,$$

$$\int_0^{\infty} x^{\alpha-1} e^{-x \cos \theta} \sin(x \sin \theta) dx = \Gamma(\alpha) \sin \alpha \theta.$$

53 计算下列积分: (1) $\int_{-1}^1 (1-x)^p (1+x)^q dx$, $\operatorname{Re} p > -1$, $\operatorname{Re} q > -1$, $\int_{-1}^1 (1-x^2)^n dx$;

$$(2) \int_0^{\pi/2} \tan^{\alpha} \theta d\theta, \int_0^{\pi/2} \cot^{\alpha} \theta d\theta, \quad |\alpha| < 1.$$

19.5)

(1) 作代换 $1+x=2u$, 即 $(1-x)=2(1-u)$, 则

$$\int_{-1}^1 (1-x)^p (1+x)^q dx = 2^{p+q+1} \int_0^1 (1-u)^p u^q du = 2^{p+q+1} B(p+1, q+1).$$

$$\text{作代换 } x^2 = u, \text{ 则 } \int_{-1}^1 (1-x^2)^n dx = 2 \int_0^1 (1-x^2)^n dx = \int_0^1 (1-u)^n u^{-1/2} du = B\left(n+1, \frac{1}{2}\right).$$

(2) 作代换 $\tan^2 \theta = x$, 则

$$\int_0^{\pi/2} \tan^{\alpha} \theta d\theta = \frac{1}{2} \int_0^{\infty} x^{\frac{\alpha}{2}} \frac{1}{(1+x)\sqrt{x}} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{\frac{\alpha-1}{2}}}{(1+x)} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{\frac{1+\alpha}{2}-1}}{(1+x)^{\frac{1-\alpha}{2}+\frac{1+\alpha}{2}}} dx$$

$$\text{由上题第 (5) 小题, 上式} = \frac{1}{2} B\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right) = \frac{1}{2} B\left(1-\frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right) = \frac{\pi}{2 \sin \pi \left(\frac{1-\alpha}{2}\right)}$$

$$= \frac{\pi}{2 \cos \frac{\alpha\pi}{2}}.$$

$$\text{作代换 } \varphi = \frac{\pi}{2} - \theta \text{ 可得 } \int_0^{\pi/2} \tan^{\alpha} \theta d\theta = \int_0^{\pi/2} \cot^{\alpha} \varphi d\varphi.$$