

一、

1) 证: $\alpha_1, \alpha_2 \in V$

$$\alpha_1 = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} \alpha_2 = \begin{pmatrix} y_1 & y_2 + iy_3 \\ y_2 - iy_3 & -y_1 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 = \begin{pmatrix} x_1 + y_1 & (x_2 + y_2) + i(x_3 + y_3) \\ (y_2 + x_2) - i(x_3 + y_3) & -(x_1 + y_1) \end{pmatrix} \in V$$

$$k\alpha_1 = \begin{pmatrix} kx_1 & kx_2 + ikx_3 \\ kx_2 - ikx_3 & -kx_1 \end{pmatrix} \in V$$

故是 - 线性空间

2) 维 $(V) = 3$

其中 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 为一组基.

$$\forall A = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

且 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 线性无关

故为一组基, 且 维 $(V) = 3$

二、

解: ① $L(\alpha_1, \alpha_2, \alpha_3) + L(\beta_1, \beta_2, \beta_3) = L(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

$$A = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & -2 & 3 \\ 0 & -1 & 1 & 0 & 2 & 1 \\ 2 & 3 & -2 & -6 & 4 & -5 \end{pmatrix}$$

$$A \rightarrow \begin{pmatrix} 1 & 0 & 0 & 10 & 2 \\ 0 & 1 & 0 & -6 & -1 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

维 $(L(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)) = \text{秩}(A) = 4$

$\alpha_1, \alpha_2, \alpha_3, \beta_1$ 为一组基

$$\textcircled{2} B = (\alpha_1, \alpha_2, \alpha_3) \quad C = (\beta_1, \beta_2, \beta_3)$$

维 $(V_1) = \text{秩}(B) = 3$, 维 $(V_2) = \text{秩}(C) = 3$

维 $(V_1 \cap V_2) = \text{维}(V_1) + \text{维}(V_2) - \text{维}(V_1 + V_2) = 2$

设 $\xi \in V_1 \cap V_2$

$$\xi = x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3 = y_1 \beta_1 + y_2 \beta_2 + y_3 \beta_3$$

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -2 \\ 1 & 1 & 2 & -2 & 2 & -3 \\ 0 & -1 & 1 & 0 & -2 & -1 \\ 2 & 3 & -2 & 6 & -4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -10 & -2 \\ 0 & 1 & 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

基础解系为: $(10, -6, -4, -1, 1, 0), (2, -1, 0, -1, 0, 1)$

$$\xi_1 = (10\alpha_1 - 6\alpha_2 - 4\alpha_3)^T = (10, -4, 2, 10)^T$$

$$\xi_2 = (2\alpha_1 - \alpha_2)^T = (1, 1, 1, 1)^T$$

ξ_1, ξ_2 为一组基

三、(1) 证: 知 $P[x]$ 的组基 $1, x, x^2$

$$(f_1(x), f_2(x), f_3(x)) = (1, x, x^2) \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (1, x, x^2) A$$

$$(g_1(x), g_2(x), g_3(x)) = (1, x, x^2) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = (1, x, x^2) B$$

$$r(A) = r(B) = 3$$

故 $f_1(x), f_2(x), f_3(x)$ 为基, $g_1(x), g_2(x), g_3(x)$ 为基

$$(2) (g_1(x), g_2(x), g_3(x)) = (f_1(x), f_2(x), f_3(x)) A^{-1} B$$

$$A^{-1} B = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$$(3) f(x) = 2 + 5x + 3x^2 = (1, x, x^2) \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= (g_1(x), g_2(x), g_3(x)) B^{-1} \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$$= (g_1(x), g_2(x), g_3(x)) \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 2 \end{pmatrix}$$

四、 1) 解: $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3) B$

$$\Delta(\beta_1, \beta_2, \beta_3) = B^{-1} \Delta(\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} -\frac{1}{7} & -\frac{3}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{6}{7} & \frac{1}{7} \\ \frac{2}{7} & -\frac{1}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 5 & 7 & 5 \\ 0 & 4 & -1 \\ 4 & 11 & 2 \end{pmatrix} \\ = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & -2 \\ 2 & 3 & -1 \end{pmatrix}$$

2) 解: $\alpha = (\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\Delta(\alpha) = (\beta_1, \beta_2, \beta_3) B^{-1} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ = (\beta_1, \beta_2, \beta_3) \begin{pmatrix} -\frac{8}{7} \\ \frac{23}{7} \\ \frac{2}{7} \end{pmatrix}$$

五、1) 解: $f(x_1, x_2, x_3) \rightarrow \begin{pmatrix} 1 & -2 & b \\ -2 & a & 4 \\ b & 4 & -2 \end{pmatrix} = A$

$f(y_1, y_2, y_3) \rightarrow \begin{pmatrix} -7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = B$

$|A| = |B| \Rightarrow -8 - 2a + 16b - ab^2 = -28$

$\text{Tr}(A) = \text{Tr}(B)$ 故 $\begin{cases} a = -2 \\ b = 2 \end{cases}$
且 $b < 2$

2) 解: ① 当 $\lambda = -7$ 时 $|\lambda E - A| = 0$

解得其基础解系: $\alpha_1 = (-1, -2, 2)$

$\eta_1 = \frac{\alpha_1}{\sqrt{|\alpha_1|}} = \frac{1}{3}(-1, -2, 2)$

② 当 $\lambda = 2$ 时, $|\lambda E - A| = 0$

基础解系: $\alpha_2 = (2, 0, 1)$, $\alpha_3 = (-2, 1, 0)$

$\beta_2 = (2, 0, 1)$, $\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$
 $= (-\frac{2}{5}, 1, \frac{4}{5})$

$\eta_2 = \frac{1}{\sqrt{5}}(2, 0, 1)$ $\eta_3 = (-\frac{2}{3\sqrt{5}}, \frac{\sqrt{5}}{3}, \frac{4}{3\sqrt{5}})$

取 $X = \begin{pmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{1}{3} & 0 & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{pmatrix} Y$

六、证: 由 $(f(x), g(x)) = 1$ 知 $\exists u(x), v(x)$ 使 $u(x)f(x) + v(x)g(x) = 1$

$$\text{故 } u(A)f(A) + v(A)g(A) = E$$

$$\text{对 } \forall \xi \in V \text{ 有 } \xi = u(A)f(A)\xi + v(A)g(A)\xi$$

$$\text{由 } u(A)f(A) + v(A)g(A) = E, \text{ 知 } u(A)f(A)\xi = \xi \text{ 或 } v(A)g(A)\xi = \xi$$

$$\text{所以 } u(A)f(A)\xi \in \ker(g(A))$$

$$v(A)g(A)\xi \in \ker(f(A))$$

$$\forall \xi \in \ker(f(A) + \ker(g(A)))$$

$$f(A)\xi = 0 \quad g(A)\xi = 0$$

$$\text{则 } \ker(f(A)) \cap \ker(g(A)) = \{0\}$$

$$\text{则 } \ker(f(A)g(A)) = \ker(f(A)) \oplus \ker(g(A))$$

七、证: 因为 A 正定, 所以存在可逆矩阵 X 使 $X^T A X = E$

对实对称矩阵 $X^T B X$, 存在正交矩阵 Y

有 $Y^T (X^T B X) Y$ 为对角

$$Y^T (X^T B X) Y = (XY)^T B (XY) = Y^T Y = E$$

故令 $P = XY$ 即满足