
CHAPTER 1

2023-2024 学年微积分（一）（上）期中考试

1 基本计算题 (每小题 6 分, 共 60 分)

1. 求极限 $l = \lim_{n \rightarrow \infty} (-1)^n \sin(\sqrt{n^2 + 1}\pi)$.

2. 求极限 $l = \lim_{x \rightarrow 0} \frac{e^{\tan x} - e^x}{x - \sin x}$.

3. 当 $x \rightarrow 0$ 时, 设 $u = \sqrt{1 + \tan x} - \sqrt{1 + \sin x}$ 的主部为 cx^k , 试确定常数 c, k 的值.

4. 求极限 $l = \lim_{x \rightarrow 0} \left(\frac{\arcsin x}{x} \right)^{\frac{1}{x^2}}$.

5. 设函数 $f(x) = \frac{x - x^2}{\sin \pi x}$, 指出 $f(x)$ 的间断点并判断其类型.

6. 设 $y = \sqrt{x \sin x \sqrt{e^x - 1}}$ ($0 < x < \pi$), 求导数 y' .

7. 设 $f(x) = \begin{cases} a + x + \sqrt{1 - x}, & x < 0 \\ 1 + b \arctan x, & x \geq 0 \end{cases}$, 求常数 a, b , 使 $f(x)$ 在 $x = 0$ 处可导.

8. 设函数 $y = y(x)$ 由方程 $y = 2e^y \sin x - 7x$ 确定的可微函数, 求 $dy|_{x=0}$.

9. 设 $y = \frac{1}{2 - 3x - 2x^2}$, 求 $y^{(n)}(0)$.

10. 设 $y = \sin^4 x - \cos^4 x$, 求 y' 及 y'' .

2 综合题 (每小题 6 分, 共 30 分)

11. 设函数 $y = y(x)$ 由参数方程 $\begin{cases} x = t^2 + 1, \\ y = 4t - t^2 \end{cases}, t > 0$ 确定, 过点 $(-1, 0)$ 作 $y = y(x)$ 的切线, 求切点 (x_0, y_0) , 并写出该切线的方程.

12. 设 $y = g(x)$ 是 $y = f(x)$ 的反函数, $f(x) = x + (1 + x)^x$, 求 $y = g(2 + x^2)$ 在 $x = 1$ 处的导数.

13. 设 $b_i \geq 0 (i = 1, 2, \dots, n, \dots)$, $x_n = \frac{b_1}{1 + b_1} + \frac{b_2}{(1 + b_1)(1 + b_2)} + \dots + \frac{b_n}{(1 + b_1)(1 + b_2) \cdots (1 + b_n)}$, 研究数列 $\{x_n\}$ 的极限的存在性.

14. 设函数 $f(x) = \begin{cases} \frac{g(x) - e^{-x}}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$, 其中 $g(x)$ 在 $(-\infty, +\infty)$ 内具有一阶连续的导数, 且 $g''(0)$ 存在, $g(0) = 1$, $g'(0) = -1$, 求 $f'(x)$, 并讨论 $f'(x)$ 在 $x = 0$ 处的连续性.

15. 一观察者站在地面上用望远镜观察一架飞机, 该飞机的高度为 20km, 正以 16km/min 的速度向观察者水平地飞过来, 试问当飞机离观察者的水平距离为 40km 时, 望远镜视角改变的速率是多少?

3 证明题 (每小题 5 分, 共 10 分)

16. 设函数 $f(x)$ 在 (a, b) 内连续, 且 $x_i \in (a, b), i = 1, 2, \dots, n$, 证明至少存在一点 $\xi \in (a, b)$, 使

$$f(\xi) = \frac{1}{n} (f(x_1) + f(x_2) + \dots + f(x_n)).$$

17. 设函数 $f(x)$ 在 $[0, 1]$ 上连续, 在 $(0, 1)$ 内可导, 并且 $f(0) = 0, f(1) = \frac{1}{2}$, 证明存在 $\xi, \eta \in (0, 1)$, 且 $\xi \neq \eta$, 使得 $f'(\xi) + f'(\eta) = \xi + \eta$.

CHAPTER 2

2023-2024 学年微积分（一）（上）期中考试参考答案

1 基本计算题 (每小题 6 分, 共 60 分)

1. Solution.

$$\begin{aligned} l &= \lim_{n \rightarrow \infty} (-1)^n n \sin \left((\sqrt{n^2 + 1} - n)\pi + n\pi \right) \\ &= \lim_{n \rightarrow \infty} n \sin \left((\sqrt{n^2 + 1} - n)\pi \right) = \lim_{n \rightarrow \infty} n \sin \left(\frac{\pi}{\sqrt{n^2 + 1} + n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1} + n} \pi = \frac{\pi}{2}. \end{aligned}$$

2. Solution.

法一.

$$\begin{aligned} l &= \lim_{x \rightarrow 0} \frac{e^x(e^{\tan x - x} - 1)}{x - \sin x} = \lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\tan^2 x}{\frac{1}{2}x^2} = 2. \end{aligned}$$

法二.

$$\begin{aligned} l &= \lim_{x \rightarrow 0} e^{\xi} \frac{\tan x - x}{x - \sin x} \quad (\xi \text{ 介于 } x \text{ 与 } \tan x \text{ 之间}) \\ &= \lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{\tan^2 x}{\frac{1}{2}x^2} = 2. \end{aligned}$$

3. Solution.

法一.

$$\begin{aligned} u &= \frac{1}{\sqrt{1 + \tan x} + \sqrt{1 + \sin x}} (\tan x - \sin x) \\ &\sim \frac{1}{2} (\tan x - \sin x) = \frac{1}{2} \tan x (1 - \cos x) \sim \frac{1}{4} x^3, \end{aligned}$$

即 $k = 3, c = \frac{1}{4}$.

法二. 由题意得 $\lim_{x \rightarrow 0} \frac{u}{cx^k} = 1$. 而

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{u}{cx^k} &= \frac{\tan x - \sin x}{cx^k(\sqrt{1+\tan x} + \sqrt{1+\sin x})} \\ &= \lim_{x \rightarrow 0} \frac{\tan x(1 - \cos x)}{2cx^k} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^3}{2cx^k} = \lim_{x \rightarrow 0} \frac{x^{3-k}}{4c},\end{aligned}$$

故 $3-k=0, 4c=1$, 即 $k=3, c=\frac{1}{4}$.

4. **Solution.**

$$\begin{aligned}l &= \lim_{x \rightarrow 0} \exp \left[\frac{1}{x^2} \ln \left(\frac{\arcsin x}{x} \right) \right] = \exp \left[\lim_{x \rightarrow 0} \frac{1}{x^2} \ln \left(\frac{\arcsin x}{x} \right) \right] \\ &= \exp \left[\lim_{x \rightarrow 0} \frac{1}{x^2} \left(\frac{\arcsin x}{x} - 1 \right) \right] = \exp \left[\lim_{x \rightarrow 0} \frac{\arcsin x - x}{x^3} \right] \\ &= \exp \left[\lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2} \right] = \exp \left[\lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x^2}}{3x^2\sqrt{1-x^2}} \right] \\ &= \exp \left[\lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{3x^2\sqrt{1-x^2}} \right] = e^{\frac{1}{6}}.\end{aligned}$$

5. **Solution.**

当 $\sin \pi x = 0$ 即 $x = k, k = 0, \pm 1, \pm 2, \dots$ 时, $f(x)$ 无定义,

故 $x = k, k = 0, \pm 1, \pm 2, \dots$ 是 $f(x)$ 的间断点.

因

$$\begin{aligned}\lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{x - x^2}{\sin \pi x} = \lim_{x \rightarrow 0} \frac{\pi x}{\sin \pi x} \cdot \frac{1-x}{\pi} = \frac{1}{\pi}; \\ \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{\pi(1-x)}{\sin \pi(1-x)} \cdot \frac{x}{\pi} = \frac{1}{\pi},\end{aligned}$$

所以 $x = 0, 1$ 是可去间断点;

当 $k = -1, \pm 2, \pm 3, \dots$ 时, $\lim_{x \rightarrow k} f(x) = \infty$, 所以 $x = -1, \pm 2, \pm 3, \dots$ 是无穷间断点.

6. **Solution.** 因

$$\ln y = \frac{1}{2} \left(\ln x + \ln \sin x + \frac{1}{2} \ln(e^x - 1) \right),$$

所以

$$y' = \frac{1}{2} \sqrt{x \sin x \sqrt{e^x - 1}} \left(\frac{1}{x} + \cot x + \frac{e^x}{2(e^x - 1)} \right).$$

7. **Solution.** 要使得 $f(x)$ 在 $x = 0$ 处可导, 必须 $f(x)$ 在 $x = 0$ 处连续, 即

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0).$$

因 $f(0) = 1$, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1 + b \arctan x) = 1 = f(0)$,

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (a + x + \sqrt{1-x}) = a + 1$, 所以 $a + 1 = 1$, 即 $a = 0$.

又 $f'_-(0) = \lim_{x \rightarrow 0} \frac{(x + \sqrt{1-x}) - 1}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{1-x}(1 - \sqrt{1-x})}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{1-x} \cdot \frac{1}{2}x}{x} = \frac{1}{2}$,

$$f'_+(0) = \lim_{x \rightarrow 0} \frac{1 + b \arctan x - 1}{x} = \lim_{x \rightarrow 0} \frac{b \arctan x}{x} = b,$$

又必须 $f'_-(0) = f'_+(0)$, 所以 $b = \frac{1}{2}$.

8. Solution.

法一. 当 $x = 0$ 时 $y = 0$.

方程两边求导

$$y' = 2e^y y' \sin x + 2e^y \cos x - 7,$$

即

$$y' = \frac{2e^y \cos x - 7}{1 - 2e^y \sin x},$$

所以

$$y'|_{x=0} = -5, \quad dy|_{x=0} = y'|_{x=0} dx = -5 dx.$$

法二. 当 $x = 0$ 时 $y = 0$.

方程两边求微分, 得

$$dy = 2e^y \sin x dy + 2e^y \cos x dx - 7 dx,$$

从而

$$dy = \frac{2e^y \cos x - 7}{1 - 2e^y \sin x} dx, \quad dy|_{x=0} = \frac{2e^y \cos x - 7}{1 - 2e^y \sin x} \Big|_{x=0} dx = -5 dx.$$

9. **Solution.** $y = \frac{1}{(1-2x)(x+2)} = \frac{1}{5} \cdot \frac{1}{x+2} - \frac{2}{5} \cdot \frac{1}{2x-1}.$

由公式 $\left(\frac{1}{(1+x)}\right)^{(n)} = \frac{(-1)^n n!}{(1+x)^{n+1}}$, 得

$$\begin{aligned} y^{(n)}(x) &= \frac{1}{5} \left(\frac{1}{x+2}\right)^{(n)} - \frac{2}{5} \left(\frac{1}{2x-1}\right)^{(n)} = \frac{1}{5} \frac{(-1)^n \cdot n!}{(x+2)^{n+1}} - \frac{2}{5} \frac{(-1)^n \cdot 2^n \cdot n!}{(2x-1)^{n+1}} \\ &= \frac{n!}{5} \left(\frac{(-1)^n}{(x+2)^{n+1}} - \frac{(-1)^n 2^{n+1}}{(2x-1)^{n+1}} \right), \end{aligned}$$

所以 $y^{(n)}(0) = \frac{n!}{5} \left(\frac{(-1)^n}{2^{n+1}} + 2^{n+1} \right).$

10. Solution.

因为

$$y = \sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = -\cos 2x,$$

所以

$$y' = 2 \sin 2x, \quad y'' = 4 \cos 2x.$$

2 综合题 (每小题 6 分, 共 30 分)

11. **Solution.** 设切点 (x_0, y_0) 对应的参数为 $t = t_0$, 则 $\frac{dy}{dx} \Big|_{t=t_0} = \frac{4-2t}{2t} \Big|_{t=t_0} = \frac{2-t_0}{t_0},$

曲线在 (x_0, y_0) 的切线方程为 $y - y_0 = \frac{2-t_0}{t_0}(x - x_0),$

即 $y - (4t_0 - t_0^2) = \frac{2-t_0}{t_0}(x - t_0^2 - 1)$ 化简得 $y = \frac{2-t_0}{t_0}x + 2t_0 - \frac{2-t_0}{t_0}$,

切线过 $(-1, 0)$, 故代入方程得 $t_0^2 - t_0 - 2 = 0$, 解得: $t_0 = -2$ (不合题意), $t_0 = 1$,

由 $t_0 = 1$ 得 $(x_0, y_0) = (2, 3)$, 故所求切线方程为 $y = x + 1$.

12. **Solution.** 由题意得 $f'(x) = 1 + (1+x)^x \left(\ln(1+x) + \frac{x}{1+x} \right)$,

所以 $f'(1) = 2(\ln 2 + 1)$.

又

$$f(1) = 3, \quad g'(3) = \frac{1}{f'(1)} = \frac{1}{2(\ln 2 + 1)},$$

因此

$$y' = g'(2+x^2) \cdot 2x = 2xg'(2+x^2), \quad y'|_{x=1} = g'(3) \cdot 2 = \frac{1}{\ln 2 + 1}.$$

13. **Solution.**

$$\begin{aligned} x_{n+1} &= \frac{b_1}{1+b_1} + \frac{b_2}{(1+b_1)(1+b_2)} + \cdots + \frac{b_n}{(1+b_1)(1+b_2)\cdots(1+b_n)} + \frac{b_{n+1}}{(1+b_1)(1+b_2)\cdots(1+b_{n+1})} \\ &= x_n + \frac{b_{n+1}}{(1+b_1)(1+b_2)\cdots(1+b_{n+1})} \geq x_n, \end{aligned}$$

故数列 $\{x_n\}$ 单调递增. 因

$$\frac{b_k}{(1+b_1)(1+b_2)\cdots(1+b_k)} = \frac{1}{(1+b_1)(1+b_2)\cdots(1+b_{k-1})} - \frac{1}{(1+b_1)(1+b_2)\cdots(1+b_k)},$$

所以

$$\begin{aligned} x_n &= 1 - \frac{1}{1+b_1} + \left(\frac{1}{1+b_1} - \frac{1}{(1+b_1)(1+b_2)} \right) + \cdots \\ &\quad + \left(\frac{1}{(1+b_1)(1+b_2)\cdots(1+b_{n-1})} - \frac{1}{(1+b_1)(1+b_2)\cdots(1+b_n)} \right) \\ &= 1 - \frac{1}{(1+b_1)(1+b_2)\cdots(1+b_n)} < 1. \end{aligned}$$

从而数列 $\{x_n\}$ 有上界, 由单调有界准则知数列得极限存在.

14. **Solution.**

$$\text{当 } x \neq 0 \text{ 时, } f'(x) = \frac{x(g'(x) + e^{-x}) - g(x) + e^{-x}}{x^2} = \frac{xg'(x) + (x+1)e^{-x} - g(x)}{x^2}.$$

当 $x = 0$ 时,

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{\frac{g(x) - e^{-x}}{x} - 0}{x} = \lim_{x \rightarrow 0} \frac{g(x) - e^{-x}}{x^2} = \lim_{x \rightarrow 0} \frac{g'(x) + e^{-x}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{g'(x) + 1 + e^{-x} - 1}{2x} = \lim_{x \rightarrow 0} \frac{g'(x) - g'(0) + e^{-x} - 1}{2x} = \frac{1}{2}(g''(0) - 1), \end{aligned}$$

因

$$\begin{aligned} \lim_{x \rightarrow 0} f'(x) &= \lim_{x \rightarrow 0} \frac{x(g'(x) + e^{-x})}{x^2} = \lim_{x \rightarrow 0} \left(\frac{g'(x) + e^{-x}}{x} - \frac{g(x) - e^{-x}}{x^2} \right) \\ &= \lim_{x \rightarrow 0} \frac{g'(x) + e^{-x}}{x} = - \lim_{x \rightarrow 0} \frac{g(x) - e^{-x}}{x^2} \\ &= g''(0) - 1 - \frac{1}{2}(g''(0) - 1) = \frac{1}{2}(g''(0) - 1) = f'(0), \end{aligned}$$

故 $f'(x)$ 在 $x = 0$ 处连续.

15. **Solution.** 设时刻 t 飞机离观察者的水平距离为 x km, 望远镜视角为 θ 弧度, 则有

$$\frac{dx}{dt} = -16 \text{ km/min}, \quad x = 20 \tan \theta,$$

故 $\theta = \arctan \frac{x}{20}$, 于是 $\frac{d\theta}{dt} = \frac{1}{20 + \frac{x^2}{20}} \frac{dx}{dt}$.

当 $x = 40$ km, $\frac{dx}{dt} = -16$ (km/min) 时, $\frac{d\theta}{dt} = \frac{1}{20 + \frac{40^2}{20}} (-16) = -\frac{4}{25} \text{ rad/min}$.

即望远镜视角改变的速率是 $-\frac{4}{25} \text{ rad/min}$.

3 证明题 (每小题 5 分, 共 10 分)

16. **Proof.** 不妨设 $a < x_1 < x_2 < \cdots < x_n < b$, 显然 $f(x)$ 在 $[x_1, x_n]$ 上连续.

故 $f(x)$ 在 $[x_1, x_n]$ 上必取得最大值 M 和最小值 m , 即 $m \leq f(x) \leq M, \forall x \in [x_1, x_n]$,

从而有 $m \leq f(x_i) \leq M, i = 1, 2, \cdots, n$, 故

$$m \leq \frac{1}{n} (f(x_1) + f(x_2) + \cdots + f(x_n)) \leq M.$$

由介值定理知至少存在一点 $\xi \in [x_1, x_n] \subset (a, b)$ 使得

$$f(\xi) = \frac{1}{n} (f(x_1) + f(x_2) + \cdots + f(x_n)).$$

17. **Proof.**

法一. 令 $F(x) = f(x) - \frac{1}{2}x^2$, 则 $F(x)$ 在 $[0, 1]$ 上连续, 在 $(0, 1)$ 内可导,

且 $F(0) = f(0) = 0, F(1) = f(1) - \frac{1}{2} = 0$.

函数 $F(x)$ 在 $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]$ 上分别应用 Lagrange 中值定理,

$\exists \eta \in \left(0, \frac{1}{2}\right), \xi \in \left(\frac{1}{2}, 1\right)$, 使得

$$\begin{aligned} F\left(\frac{1}{2}\right) - F(0) &= F'(\eta) \frac{1}{2}, & \text{即 } 2F\left(\frac{1}{2}\right) &= f'(\eta) - \eta, \\ F(1) - F\left(\frac{1}{2}\right) &= F'(\xi) \frac{1}{2}, & \text{即 } -2F\left(\frac{1}{2}\right) &= f'(\xi) - \xi, \end{aligned}$$

两式相加得 $f'(\eta) - \eta + f'(\xi) - \xi = 0$, 即 $f'(\xi) + f'(\eta) = \xi + \eta$.

法二. 令 $G(x) = f(x) - f(1-x)$, 则

$$G'(x) = f'(x) + f'(1-x), \quad G(0) = f(0) - f(1) = -\frac{1}{2}, \quad G\left(\frac{1}{2}\right) = 0.$$

应用 Lagrange 中值定理得, $\exists \xi \in \left(0, \frac{1}{2}\right)$, 使

$$G\left(\frac{1}{2}\right) - G(0) = G'(\xi) \left(\frac{1}{2} - 0\right) \quad \text{即} \quad f'(\xi) + f'(1-\xi) = 1.$$

令 $\eta = 1 - \xi \in \left(\frac{1}{2}, 1\right)$, 即得 $f'(\xi) + f'(\eta) = \xi + \eta$.