23-24

一、1 D  A $x_n=\begin{cases} n & n奇 \\ 0 & n偶 \end{cases}$ $y_n=\begin{cases} 0 & n奇 \\ n & n偶 \end{cases}$ $\lim\limits_{n\to\infty} x_n y_n = 0$

B $x_n=\begin{cases} n & n奇 \\ 1 & n偶 \end{cases}$ $y_n=\begin{cases} \frac{1}{n} & n奇 \\ n & n偶 \end{cases}$ $\lim\limits_{n\to\infty} x_n y_n = \infty$

C. n·有 有界

D ✓

2 D

$\lim\limits_{x\to 0^+} e^{\overset{\to -\infty}{-\frac{1}{x^2}}} \overset{\to\frac{\pi}{2}}{\arctan\frac{1}{x}} = 0$

$\lim\limits_{x\to 0^-} e^{\overset{\to -\infty}{-\frac{1}{x^2}}} \overset{\to -\frac{\pi}{2}}{\arctan\frac{1}{x}} = 0$

3 A

4. C

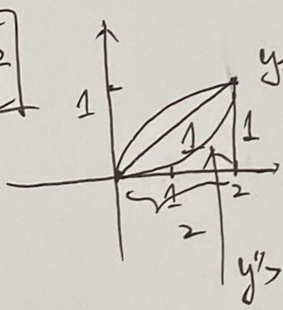
$f(x)=\int e^{-x}+\cos x\,dx$

$= -e^{-x}+\sin x + C_1$

$F(x)=\int -e^{-x}+\sin x + C_1\,dx$

$= e^{-x}-\cos x + C_1 x + C_2$

5 C




$S_1 < 1 < S_2$

6 2

7. 3 $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=3$  $\Delta y \sim 3\Delta x$

$f(\frac{1}{n}) \sim \frac{3}{n}$

8. −2 $e^{2x+y}-\cos(xy)=e-1$  $e^{y}-\cos 0 = e-1 \Rightarrow y=1$

$e^{2x+y}(2+y')+\sin(xy)(y+xy')=0$

$e(2+y')=0 \Rightarrow y'=-2$

9. $y=\frac{x^2+x+1}{\sqrt{x^2-1}}$ $(x>1)$

斜渐近线

$\lim\limits_{x\to+\infty}\frac{y}{x}=\lim\limits_{x\to+\infty}\frac{x^2+x+1}{x\sqrt{x^2-1}}=1$

$\lim\limits_{x\to+\infty}\left(\frac{x^2+x+1}{\sqrt{x^2-1}}-x\right)$

$=\lim\limits_{x\to+\infty}\frac{x^2+x+1-x\sqrt{x^2-1}}{\sqrt{x^2-1}}=\lim\limits_{x\to+\infty}\frac{(x^2+x+1)(\sqrt{x^2-1})-x^3+x}{x^2-1}$

$=1$

$y=x+1$

10. $y=\ln\cos x$  $y'=\frac{1}{\cos x}(-\sin x)$

$s=\int_0^{\frac{\pi}{4}}\sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\,dx$

$=\int_0^{\frac{\pi}{4}}\sqrt{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}\,dx$

$=\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2 x}\,dx$

$=\int_0^{\frac{\pi}{4}}\sec x\,dx$

$=\int_0^{\frac{\pi}{4}}\frac{\cos x}{\cos^2 x}\,dx$

$=\int_0^{\frac{\pi}{4}}\frac{d\sin x}{1-\sin^2 x}$

$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\,d\sin x$

$=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|\Big|_0^{\frac{\pi}{4}}$

$=\frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right)$

要

$=\frac{1}{2}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}$

$=\frac{1}{2}\ln\frac{(2+\sqrt{2})^2}{4-2}$

$=\frac{1}{2}\ln\frac{6+4\sqrt{2}}{2}$

$=\frac{1}{2}\ln(3+2\sqrt{2})$

$(1+\sqrt{2})^2 = 3+2\sqrt{2}$

11. $\lim\limits_{x\to0^+} x^{\frac{1}{\ln(e^x-1)}}$

$= \lim\limits_{x\to0^+} e^{\frac{1}{\ln(e^x-1)}\ln x} = e^1 = e$

~~lim~~

$\lim\limits_{x\to0^+} \dfrac{\ln x}{\ln(e^x-1)}$

$= \lim\limits_{x\to0^+} \dfrac{\frac{1}{x}}{\frac{1}{e^x-1}e^x}$

$= \lim\limits_{x\to0^+} \dfrac{e^x-1}{x}$

$= \lim\limits_{x\to0^+} \dfrac{e^x}{1} = 1$

12. $y - x = e^{x(1-y)}$

$y' - 1 = e^{x(1-y)}(1-y \cancel{+} x(-y'))$

$= e^{x(1-y)}(1-y-xy')$

$y'-1 = e^{x(1-y)}(1-y) - xe^{x(1-y)}y'$

$(1+xe^{x(1-y)})y' = e^{x(1-y)}(1-y)+1$

$y' = \dfrac{1+(1-y)e^{x(1-y)}}{1+xe^{x(1-y)}}$

$\cancel{\lim\limits_{x\to0} y'|_{x=0}}$

$y|_{x=0} - 0 = e^0 = 1$ $\quad f'(0) = \frac{1}{1} = 1$

$\lim\limits_{x\to0} \dfrac{f(x)-1}{x} = \lim\limits_{x\to0} f'(x)$

13. $\dfrac{dy}{dx} - \dfrac{2}{x+1}y = (x+1)^3 \qquad y(0) = \frac{1}{2}$

- $\dfrac{dy}{dx} - \dfrac{2}{x+1}y = 0$

$\dfrac{dy}{y} = 2\dfrac{dx}{x+1}$

$\ln|y| = 2\ln|x+1| + C_1$

$y = C(x+1)^2$

- $y = C(x)(x+1)^2$

$\dfrac{dy}{dx} = C'(x)(x+1)^2 + C(x)2(x+1) = \dfrac{2}{x+1}y + (x+1)^3$

$C'(x) = x+1$

$\Rightarrow C(x) = \frac{x^2}{2} + x + \tilde{C}$

[illegible] $y = \left(\frac{x^2}{2} + x + \tilde{C}\right)(x+1)^2$

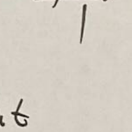
$y(0) = \frac{1}{2}$

$y(0) = \tilde{C} = \frac{1}{2}$

$y = \left(\frac{x^2}{2} + x + \frac{1}{2}\right)(x+1)^2$

$= \dfrac{(x+1)^4}{2}$

14. $\displaystyle\int \frac{1}{x^4\sqrt{1+x^2}}dx$




$\underline{\underline{x=\tan t}} \displaystyle\int \frac{1}{\tan^4 t \sec t}\sec^2 t\, dt$

$= \displaystyle\int \frac{\cos^4 t}{\sin^4 t}\frac{1}{\cos t}dt$

$= \displaystyle\int \frac{\cos^3 t}{\sin^4 t}dt$

$= \displaystyle\int \frac{1-\sin^2 t}{\sin^4 t}d\sin t$

$= \dfrac{\sin^{-4+1} t}{-4+1} - \dfrac{\sin^{-2+1} t}{-2+1} + C$

$= -\dfrac{1}{3}\dfrac{1}{\sin^3 t} + \dfrac{1}{\sin t} + C$

$= -\dfrac{1}{3}\left(\dfrac{\sqrt{1+x^2}}{x}\right)^3 + \dfrac{\sqrt{1+x^2}}{x} + C$

15. $\int_0^{\pi^2} \sqrt{x}\cos\sqrt{x}\,dx$

$\overset{t=\sqrt{x}}{\underset{x=t^2}{=}} \int_0^{\pi} t\cos t\; 2t\,dt$

$= 2\int_0^{\pi} t^2\cos t\,dt$

$= 2\int_0^{\pi} t^2\,d\sin t$

$= 2t^2\sin t\Big|_0^{\pi} - 2\int_0^{\pi}\sin t\,dt^2$

$= -4\int_0^{\pi} t\sin t\,dt$

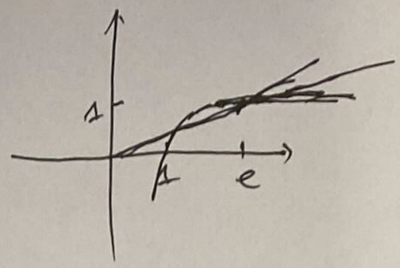
$= 4\int_0^{\pi} t\,d\cos t$

$= 4t\cos t\Big|_0^{\pi} - 4\int_0^{\pi}\cos t\,dt$

$= -4\pi - 4\int_0^{\pi} d\sin t$

$= -4\pi$

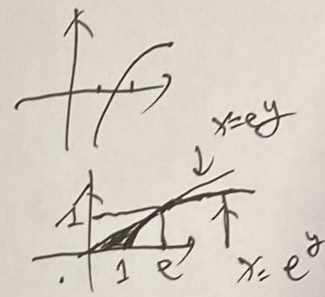
16.




$y=\ln x$

$y=\frac{x}{e}$

$\ln x=\frac{x}{e}$




(1) 16. $S_D=\int_0^1 e^y - ey\,dy$

$= e^y - e\frac{y^2}{2}\Big|_0^1$

$= e-\frac{e}{2}=\frac{e}{2}.$

(2) $V_y=\int_0^1 \pi(e^y)^2-\pi(ey)^2\,dy$

$=\pi\int_0^1 e^{2y}-e^2y^2\,dy$

$=\pi\frac{e^{2y}}{2}\Big|_0^1-\pi e^2\frac{y^3}{3}\Big|_0^1$

$=\pi\left(\frac{e^2}{2}-\frac{1}{2}-\frac{e^2}{3}\right)$

$=\pi\left(\frac{e^2}{6}-\frac{1}{2}\right)$

17. 证. $\cancel{f(x)=xe^x}$ $F(x)=f(x)e^x$

$\cancel{f'(x)=e^x}$ $F'(x)=f'(x)e^x+f(x)e^x$

$=(f'(x)+f(x))e^x \overset{?}{=} 0$

· $f(0)\,f(1)>0$

· $f(0)\,f(\frac{1}{2})<0 \Rightarrow f(\frac{1}{2})\,f(1)<0$

$\Downarrow$ $\Downarrow$

$\exists\xi_1\in(0,\frac{1}{2})$ $\exists\xi_2\in(\frac{1}{2},1)$

$f(\xi_1)=f(\xi_2)=0$

$\Rightarrow F(\xi_1)=F(\xi_2)=0$

$\Rightarrow F'(\xi)=0$

18.

$F'(x)=\lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x}$

$=\lim_{\Delta x\to 0}\frac{\frac{\int_a^{x+\Delta x}f(t)dt}{x+\Delta x-a}-\frac{\int_a^{x}f(t)dt}{x-a}}{\Delta x}$

$=\lim_{\Delta x\to 0}\frac{(x-a)\int_a^{x+\Delta x}f(t)dt-(x-a)\int_a^{x}f(t)dt-\Delta x\int_a^x f(t)dt}{(x-a)(x+\Delta x-a)(\Delta x)}$

$=\lim_{\Delta x\to 0}\frac{(x-a)\int_x^{x+\Delta x}f(t)dt-\Delta x\int_a^{x}f(t)dt}{(x-a)(x+\Delta x-a)\Delta x}$

$$\int_0^1 \frac{\ln x}{1-x^2}dx$$

$$= \int_0^{\frac{1}{2}} \frac{\ln x}{1-x^2}dx + \int_{\frac{1}{2}}^1 \frac{\ln x}{1-x^2}dx$$

$$\int_{\frac{1}{2}}^1 \frac{\ln x}{1-x^2}dx$$

$$= \int_{\frac{1}{2}}^1 \frac{\ln x}{(1-x)(1+x)}dx$$

$$\lim_{x\to 1^+} \frac{\left|\frac{\ln x}{(1-x)(1+x)}\right|}{\left|\frac{1}{(1+x)}\right|}$$

$$= \lim_{x\to 1} \left|\frac{\ln x}{1-x^2}\right| = +\infty$$

$\frac{1}{1+x}$ 发散

$$\int_{\frac{1}{2}}^1 \left|\frac{\ln x}{1-x^2}\right| dx$$

$$\le \ln\tfrac{1}{2} \int_{\frac{1}{2}}^1 \frac{1}{1-x^2}dx$$

$$= \ln\tfrac{1}{2} \int_{\frac{1}{2}}^1 \frac{\frac{1}{1-x}+\frac{1}{1+x}}{2}dx$$

$$= \tfrac{1}{2}\ln\tfrac{1}{2} \ln\left|\frac{1+x}{1-x}\right|\Big|_{\frac{1}{2}}^1$$

$$= \tfrac{1}{2}\ln$$

$$\int_0^{\frac{1}{2}} \frac{\ln x}{1-x^2}dx$$

$$\lim_{x\to 0^+} \frac{\left|\frac{\ln x}{1-x^2}\right|}{\frac{1}{\sqrt{x}}} = 0$$

$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x}}dx$ 收敛

$$\lim_{x\to a^+} F(x) = \lim_{x\to a^+} \frac{\int_a^x f(t)\,dt}{x-a}$$

$$= \lim_{x\to a^+} \frac{f(x)}{1}$$

$$= f(a)$$

18

18

$$\lim_{\Delta x\to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\frac{\int_a^{x+\Delta x} f(t)\,dt}{x+\Delta x-a} - \frac{\int_a^x f(t)\,dt}{x-a}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{(x-a)\int_a^{x+\Delta x} f(t)\,dt - (x-a)\int_a^x f(t)\,dt - \Delta x\int_a^x f(t)\,dt}{(x+\Delta x-a)(x-a)\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{(x-a)\int_x^{x+\Delta x} f(t)\,dt - \Delta x\int_a^x f(t)\,dt}{(x+\Delta x-a)(x-a)\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} f(t)\,dt}{(x+\Delta x-a)\Delta x} - \frac{\int_a^x f(t)\,dt}{(x-a)^2}$$

$(x+\Delta x-a)\Delta x$

$\Delta x\cdot x+\Delta x^2-a\Delta x$

$$= \lim_{\Delta x\to 0} \frac{(x-a)f(x+\Delta x) - \int_a^x f(t)\,dt}{(x-a)(x+2\Delta x-a)}$$

$$= \frac{(x-a)f(x) - \int_a^x f(t)\,dt}{(x-a)^2} = \frac{\int_a^x (f(x)-f(t))\,dt}{(x-a)^2} > 0$$

$$f(t) - f(a) = f'(\xi)(x-a)$$

$$= \cancel{(x-a)f(x) - \int_a^x f'(\xi)(x-a)}$$

$f'(x)>0 \Rightarrow f(x)\ge f(t)$